

VECTOR SPIN GLASSES WITH MATTIS INTERACTION II: NON-CONVEX HIGH-TEMPERATURE MODELS

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ABSTRACT. This paper constitutes the second part of a two-paper series devoted to the systematic study of vector spin glass models whose energy function involves a spin glass part and a general Mattis interaction part.

In this paper, we focus on models whose spin glass part *does not* satisfy the usual convexity assumption. In this case, the Parisi formula breaks down, and there are no known methods to fully identify the limit free energy. It was suggested in [44] that the limit free energy may be described using the unique solution of a partial differential equation of Hamilton–Jacobi type. In the present paper, we prove the validity of this conjecture in the high-temperature regime and provide an explicit representation for the free energy in terms of critical points.

Using the duality between the free energy and large deviation principles, one can then easily deduce from the previous result a large deviation principle for the mean magnetization as well as a representation for the free energy of spin glass models with additional Mattis interaction at high temperature.

In the companion paper [20], we establish similar results at all temperatures for models whose spin glass part is assumed to satisfy the usual convexity assumption.

KEYWORDS AND PHRASES: DISORDERED SYSTEMS, LARGE DEVIATION PRINCIPLES, SPIN GLASSES, HAMILTON–JACOBI EQUATIONS.

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1. INTRODUCTION

1.1. Preamble. This paper is the second paper of a two-part series in which we investigate general mean-field spin glass models with Mattis interaction. In this paper, we focus on models whose spin glass part does not satisfy the usual convexity assumption.¹ An important model in this class is a specific disordered variant of the so-called Restricted Boltzmann Machine, which can be naturally mapped to a bipartite Sherrington-Kirkpatrick spin glass featuring Mattis interactions. This model consists of a system of ± 1 spins interacting through the following energy function

$$(1.1) \quad E_N(\sigma) = \frac{\beta}{\sqrt{N}} \sum_{i,j=1}^N W_{ij} \sigma_{1,i} \sigma_{2,j} + \sum_{i=1}^N \chi_i^1 \sigma_{1,i} + \sum_{i=1}^N \chi_i^2 \sigma_{2,i},$$

where $\sigma = (\sigma_1, \sigma_2) \in \{-1, 1\}^N \times \{-1, 1\}^N$ is the bipartite spin configuration, and the couplings $(W_{ij})_{1 \leq i, j \leq N}$ are independent centered Gaussian random variables with variance 1, representing the quenched random weights of the network. Furthermore, the vectors $(\chi_i^1)_{1 \leq i \leq N}$ and $(\chi_i^2)_{1 \leq i \leq N}$ are i.i.d. centered ± 1 Bernoulli random variables that define the quenched Mattis interactions, acting as planted patterns for the two respective layers.

Restricted Boltzmann Machines are significant because they provide a simple yet powerful framework for unsupervised learning. In this analogy, the spins correspond to binary units, the couplings to weights, and learning amounts to shaping an energy landscape with many metastable states [54, 52]. This connection has enabled fruitful exchanges between machine learning and spin glass theory, with tools from statistical mechanics helping to analyze restricted Boltzmann machines' phase behavior, expressivity, and learning dynamics [14, 31]. Some of this work has been produced by John Hopfield and Geoffrey Hinton, to whom the Nobel committee awarded the 2024 Physics prize, citing in particular their foundational works on these models and their underlying physical analogies [1, 38, 39, 40].

In this series of papers [20, 21], our interest is twofold. First, we identify the large N limit of the free energy defined by

$$(1.2) \quad f_N = -\frac{1}{N} \log \left(\frac{1}{4^N} \sum_{\sigma \in \{-1, 1\}^N \times \{-1, 1\}^N} e^{E_N(\sigma)} \right),$$

under the assumption that β is small. Second, assuming the same condition on β , we establish large deviation principles (see Definition 1.7) for the $\mathbb{R}^2$ -valued mean magnetization

$$m_N = (m_{N,1}, m_{N,2}) = \left(\frac{1}{N} \sum_{i=1}^N \chi_i^1 \sigma_{1,i}, \quad \frac{1}{N} \sum_{i=1}^N \chi_i^2 \sigma_{2,i} \right)$$

¹By the usual convexity assumption, we mean the assumption that the function $\xi : \mathbb{R}^{D \times D} \rightarrow \mathbb{R}$ defined below in (1.6), is convex on the set of symmetric positive semi-definite matrices. For the model (1.1), we have $D = 2$ and $\xi(a) = a_{11}a_{22}$.

under the Gibbs measure $\langle \cdot \rangle_N$ defined by

$$\langle g(\sigma) \rangle_N \propto \frac{1}{4^N} \sum_{\sigma \in \{-1,1\}^N \times \{-1,1\}^N} g(\sigma) e^{E_N(\sigma)}.$$

For models without Mattis interaction that satisfy the usual convexity assumption, the free energy can be computed using the Parisi formula [49, 50]. For models such as (1.1), computing the limit free energy, even in the absence of Mattis interaction, is a difficult problem that has so far not been resolved. Under the assumption that β is small enough, this problem becomes significantly easier. For certain models it has been shown that in this regime, the system exhibits a property known as *replica symmetry* [27, 55]. In this case, the limit free energy can be linked to the solution of a finite dimensional partial differential equation [35, 10, 4, 13, 11, 8, 34]. In an attempt to tackle the computation of the free energy outside of the replica symmetric phase (i.e. at high β), a new approach has been put forward relying on an infinite dimensional partial differential equation [29, 44, 43, 46]. In this setting, one introduces an enriched version of the model depending on an additional parameter q which can be encoded as a càdlàg function $[0, 1) \rightarrow \mathbb{R}_+^2$ with increasing coordinates. Here, and throughout, we write $\mathbb{R}_+ = [0, \infty)$.

Conjecturally, the large N limit of the free energy of this enriched model is the unique solution of a partial differential equation of Hamilton–Jacobi type with time parameter $t = \beta^2/2$ and space parameter q . When one chooses q to be the constant path taking the value $0 \in \mathbb{R}_+^2$, the free energy of the enriched model is equal to the free energy of the original model, and the system is replica symmetric in the low β regime. But, when q is chosen to be a path with *strictly* increasing coordinates, the system corresponding to the enriched model is not replica symmetric, even in the low β regime². Therefore, the verification of this conjectural link between the free energy and the Hamilton–Jacobi equation is highly non-trivial, even in the low β regime. We dive deeper into this issue of replica symmetry breaking versus replica symmetry in Section 4.3.

Our main results are stated for very general models in Theorem 1.4 and Theorem 1.8. The first main contribution of this paper is a verification, in the low β regime, of this link between the free energy and the solution of an infinite dimensional PDE for a very general class of non-convex spin glass models. This yields an implicit representation for the free energy of enriched versions of those models. When the enrichment parameter is set to 0, this result reduces to a statement about the free energy of a replica symmetric system, but otherwise this is not the case. From this, using well-known tools from large deviation theory, we will identify the limit free energy of models

²The system exhibits infinitely many steps of replica symmetry breaking in the small β regime in this case, see Remark 1.12.

with Mattis interaction and establish a large deviation principle for the mean magnetization, still in the low β regime.

The application of our main results to the model in (1.1) can be summarized as follows. We let η be a standard real-valued Gaussian random variable, and χ be a centered ± 1 Bernoulli random variable. For $y \in \mathbb{R}_+^2$ and $x \in \mathbb{R}^2$, we define

$$\phi(y; x) = \sum_{i=1}^2 \left(-\mathbb{E} \log \cosh(\sqrt{2y_i} \eta + x_i \chi) + y_i \right).$$

For some β_* specified in the theorem below, we will show in Proposition 4.8 that, for every $x \in \mathbb{R}^2$, the partial differential equation

$$\begin{cases} \partial_t g(t, y; x) - \partial_{y_1} g(t, y; x) \partial_{y_2} g(t, y; x) = 0 \\ g(0, y; x) = \phi(y; x) \end{cases}$$

admits a unique differentiable solution $g(\cdot, \cdot; x)$ on $[0, \beta_*^2/2) \times \mathbb{R}_+^2$.

Theorem 1.1. *There exists $\beta_* \geq \frac{1}{2\sqrt{2}}$ such that for every $\beta < \beta_*$ and almost every realization of $(W_{i,j})_{i,j}$, $(\chi_i^1)_i$ and $(\chi_i^2)_i$, the following holds. We have*

$$(1.3) \quad \lim_{N \rightarrow +\infty} f_N = \inf_{m \in \mathbb{R}^2} \sup_{x \in \mathbb{R}^2} \left\{ g\left(\frac{\beta^2}{2}, 0; x\right) + m \cdot x - (m_1 + m_2) \right\} - \frac{\beta^2}{2},$$

and the $\mathbb{R}^2$ -valued mean magnetization m_N satisfies a large deviation principle under $\langle \cdot \rangle_N$ with rate function J defined for $m \in \mathbb{R}^2$ by

$$J(m) = -(m_1 + m_2) + g^*\left(\frac{\beta^2}{2}, 0; m\right) + \sup_{m' \in \mathbb{R}^2} \left\{ m'_1 + m'_2 - g^*\left(\frac{\beta^2}{2}, 0; m'\right) \right\},$$

where $g^*\left(\frac{\beta^2}{2}, 0; m\right) = \sup_{x \in \mathbb{R}^2} \left\{ x \cdot m + g\left(\frac{\beta^2}{2}, 0; x\right) \right\}$.

Remark 1.2. We mention that the quantity $g\left(\frac{\beta^2}{2}, 0; x\right)$ can be understood in more familiar terms using a critical point representation. For $t < \beta_*^2/2$, there is a unique solution $z^x = (z_1^x, z_2^x) \in \mathbb{R}_+^2$ to the fixed point equation

$$(1.4) \quad z = \nabla_y \phi(t(z_2, z_1); x) \quad \text{for } z \in \mathbb{R}_+^2,$$

and we have

$$(1.5) \quad g(t, 0; x) = \phi(t(z_2^x, z_1^x); x) - t z_1^x z_2^x.$$

In addition, we also have that $\nabla_y g(t, 0; x) = z^x$. □

We refer to (1.5) as a critical-point representation since z^x as a solution of (1.4) is a critical point of the function on the right-hand side of (1.5). Also, we do not expect that the lower bound $\frac{1}{2\sqrt{2}}$ for β_* is sharp.

To the best of our knowledge, with the presence of Mattis interaction, identifying the limit free energy at high temperature without additional assumptions on replica symmetry has not been achieved. However, much progress has been made in related spin glass systems and neural networks. For two-layer bipartite models, such as the analogical neural network and

standard Restricted Boltzmann Machines, the replica-symmetric (RS) ansatz has been derived using interpolation and self-averaging properties [12, 5]. These methods have been extended to multi-layer geometries like Deep Boltzmann Machines. In this context, the RS ansatz can be bounded by Sherrington-Kirkpatrick models in the annealed region [6] and expressed through minimax formulas [33]. Other deep architectures rely on transport equations to handle RS and one-step replica symmetry breaking (1RSB) [2], while exact solutions without explicit RS assumptions are limited to specific regimes like the Nishimori line [7]. Furthermore, the existence of the infinite-volume thermodynamic limit for the free energy in mean field neural networks has been rigorously established by assuming the measure-concentration of order parameters [3]. In short, computing the limit free energy in all these works requires either imposing a replica-symmetric assumption (or limited symmetry-breaking) or restricting the analysis to specific regimes where such symmetry naturally holds.

For models satisfying the usual convexity assumption, computing the free energy in the absence of Mattis interaction is a classical problem, and it is now well known that the limit of the free energy can be understood as the supremum of an explicit functional, which is the celebrated Parisi formula [51, 36, 53]. For those models, the free energy can also be studied from the perspective of Hamilton–Jacobi equations, which leads to a more systematic and general treatment. The recent developments regarding this are summarized in [45].

On the other hand, models with Mattis interaction have not yet received a similar systematic treatment. The identification of their free energy relies on model-dependent proofs. Furthermore, large deviation principles for the mean magnetization have traditionally been established by computing a constrained version of the free energy

$$f_N(A) = -\frac{1}{N} \log \left(\frac{1}{4^N} \sum_{\sigma \in A} e^{E_N(\sigma)} \right),$$

which leads to technical proofs [37, 32, 17, 50, 25]. Similar techniques have also been employed to prove large deviation principles in the context of classical spin glass models [15, 41], see also [42]. For spin glass models with additional Mattis interaction, there are no known general results that allow for a systematic identification of the free energy nor the verification of large deviation principles for the mean magnetization.

In this series of papers, we resolve this problem and establish systematic results using a new approach that relies on the computation of the following object

$$f_N^G = -\frac{1}{N} \log \left(\frac{1}{2^N} \sum_{\sigma \in \{-1,1\}^N \times \{-1,1\}^N} \exp \left(\frac{\beta}{\sqrt{N}} \sum_{i,j=1}^N W_{ij} \sigma_{1,i} \sigma_{2,j} + NG(m_N) \right) \right).$$

The main difference of our approach is that the continuous function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ encoding the Mattis interaction is now treated as a *parameter of the model* instead of being fixed. Furthermore, as opposed to the traditional approach, we do not impose constraints on the spin configuration σ . This allows for smoother, shorter, and less technical computations that one can carry out in a model-independent way.

To compute $\lim_{N \rightarrow +\infty} f_N^G$, we can proceed as follows. We let $\langle \cdot \rangle_N^G$ denote the Gibbs measure associated with f_N^G . We start with the case $G(m) = x \cdot m$. Under this assumption, the sum structure of

$$NG(m_N) = x_1 \sum_{i=1}^N \chi_i^1 \sigma_{1,i} + x_2 \sum_{i=1}^N \chi_i^2 \sigma_{2,i}$$

makes the computation of $\lim_{N \rightarrow +\infty} f_N^{m \mapsto x \cdot m}$ extremely similar to the computation of $\lim_{N \rightarrow +\infty} f_N^0$. We show that in this case the free energy can be computed, at least in the small β region, using a critical point representation for $\lim_{N \rightarrow +\infty} f_N^{m \mapsto x \cdot m}$ in the style of Remark 1.2. This representation can then be connected to the aforementioned Hamilton–Jacobi equation. Then, as a direct application of the Gärtner–Ellis theorem, we deduce from the knowledge of $\lim_{N \rightarrow +\infty} f_N^{m \mapsto x \cdot m}$ for all $x \in \mathbb{R}^2$, a large deviation principle for m_N under $\langle \cdot \rangle_N^0$ with an explicit rate function I . Varadhan’s lemma then implies that for every continuous function G ,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \log \left\langle e^{NG(m_N)} \right\rangle_N^0 = \sup_m \{G(m) - I(m)\}.$$

Observing that $\frac{1}{N} \log \left\langle e^{NG(m_N)} \right\rangle_N^0 = f_N^G - f_N^0$ it follows that $\lim_{N \rightarrow +\infty} f_N^G$ can be computed. Finally, as an application of Bryc’s inverse Varadhan lemma, we deduce from the previous display that m_N satisfies a large deviation principle under $\langle \cdot \rangle_N^G$ with an explicit rate function I^G . Finally, using $G(m) = m_1 + m_2$ we obtain Theorem 1.1. Note that here we have used crucially that the previous display holds *for all* continuous functions G .

1.2. Setting. In this section, we define the class of models that we treat.

For $m, n \in \mathbb{N}$, we denote by $\mathbb{R}^{m \times n}$ the space of all $m \times n$ real matrices. For any $a = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathbb{R}^{m \times n}$, we denote its j -th column vector as $a_{\bullet j} = (a_{ij})_{1 \leq i \leq m}$ and its i -th row vector as $a_{i \bullet} = (a_{ij})_{1 \leq j \leq n}$. If not specified, a vector is understood to be a column vector. For a matrix or vector a , we denote by $a^\top$ its transpose. For $a, b \in \mathbb{R}^{m \times n}$, we write $a \cdot b = \sum_{ij} a_{ij} b_{ij}$, $|a| = \sqrt{a \cdot a}$, and similarly for vectors.

For $n \in \mathbb{N}$, let S^n be the linear space of $n \times n$ real symmetric matrices and also let S_+^n (resp. S_{++}^n) be the subset consisting of positive semi-definite (resp. definite) matrices. For $a, b \in S^n$, we write $a \geq b$ provided $a - b \in S_+^n$, which gives a natural partial order on S^n .

We start by defining the spin glass part of the model. Let $D \in \mathbb{N}$ be the dimension of a single spin. Let P_1 be a finite measure supported on the unit

ball $\{\tau \in \mathbb{R}^D : |\tau| \leq 1\}$ of $\mathbb{R}^D$ and we assume that the linear span of vectors in $\text{supp } P_1$ is $\mathbb{R}^D$.

For each $N \in \mathbb{N}$, we denote by $\sigma = (\sigma_{\bullet i})_{i=1}^N = (\sigma_{di})_{1 \leq d \leq D, 1 \leq i \leq N} \in \mathbb{R}^{D \times N}$ the spin configuration consisting of N spins $\sigma_{\bullet i} \in \mathbb{R}^D$ understood as column vectors. We view σ as a $D \times N$ matrix. We sample each spin independently from P_1 , and thus the distribution of σ is given by

$$dP_1^{\otimes N}(\sigma) = \otimes_{i=1}^N dP_1(\sigma_{\bullet i}).$$

We let $\xi : \mathbb{R}^{D \times D} \rightarrow \mathbb{R}$ be a smooth function that can be expressed as a power series. For each N , we consider a centered Gaussian field $(H_N(\sigma))_{\sigma \in \mathbb{R}^{D \times N}}$ with covariance structure given by

$$(1.6) \quad \mathbb{E}[H_N(\sigma)H_N(\sigma')] = N\xi\left(\frac{\sigma\sigma'^T}{N}\right), \quad \forall \sigma, \sigma' \in \mathbb{R}^{D \times N}.$$

The Hamiltonian $H_N : \mathbb{R}^{D \times N} \rightarrow \mathbb{R}$ is the spin glass part of the model. For convenience we will also use the function $\theta : \mathbb{R}^{D \times D} \rightarrow \mathbb{R}$ defined by

$$(1.7) \quad \theta(a) = a \cdot \nabla \xi(a) - \xi(a), \quad \forall a \in \mathbb{R}^{D \times D}.$$

We now describe the Mattis interaction part of the model. Let $L \in \mathbb{N}$ and let $(\chi_i)_{i \in \mathbb{N}}$ be i.i.d. $\mathbb{R}^L$ -valued random vectors, which is particularly independent from $H_N(\sigma)$. Let $d \in \mathbb{N}$ and $h : \mathbb{R}^D \times \mathbb{R}^L \rightarrow \mathbb{R}^d$ be a bounded³ measurable function. For each $i \in \mathbb{N}$, we view $h(\sigma_{\bullet i}, \chi_i)$ as a generalized spin and define the mean magnetization

$$(1.8) \quad m_N = \frac{1}{N} \sum_{i=1}^N h(\sigma_{\bullet i}, \chi_i).$$

Let $G : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function, and we add to the system the Curie–Weiss-type interaction $NG(m_N)$.

At this point, we have described the spin glass part and the Mattis interaction part of our energy function. As explained in the introduction, recently a new approach has been put forward to describe the free energy of spin glass models (without Mattis interaction). In this approach, one needs to enrich the model by adding several new parameters.

To define the enriched version of the model, we need to add external fields parameterized by increasing paths and driven by Poisson–Dirichlet cascades. Let $\mathcal{Q}_\infty$ be the collection of bounded increasing càdlàg paths $p : [0, 1) \rightarrow S_+^D$, where the monotonicity is understood as $p(s) - p(s') \in S_+^D$ for $s > s'$. Throughout given $p \in \mathcal{Q}_\infty$ we will define

$$p(1) = \lim_{s \rightarrow 1^-} p(s),$$

which is well-defined by monotonicity of p .

Throughout, let $\mathfrak{R}$ be the Ruelle probability cascade (RPC) with overlap uniformly distributed over $[0, 1]$ (see [47, Theorem 2.17]). Precisely, $\mathfrak{R}$

³Throughout it will be enough to assume that h is bounded on the support of the finite measure $P_1 \otimes \text{Law}(\chi_1)$.

is a random probability measure on the unit sphere of a fixed separable Hilbert space (the exact form of the space is not important), with the inner product denoted by $\alpha \wedge \alpha'$. Let α and α' be independent samples from $\mathfrak{R}$. Then, the law of $\alpha \wedge \alpha'$ under $\mathbb{E}\mathfrak{R}^{\otimes 2}$ is the uniform distribution over $[0, 1]$, where $\mathbb{E}$ integrates the randomness of $\mathfrak{R}$. This overlap distribution uniquely determines $\mathfrak{R}$ (see [47, Theorem 2.13]). Almost surely, the support of $\mathfrak{R}$ is ultrametric in the induced topology. For rigorous definitions and basic properties, we refer to [47, Chapter 2] (also see [29, Chapter 5]). We also refer to [22, Section 4] for the construction and properties of $\mathfrak{R}$ useful in this work.

For $q \in \mathcal{Q}_\infty$ and almost every realization of $\mathfrak{R}$, let $(w^q(\alpha))_{\alpha \in \text{supp } \mathfrak{R}}$ be the $\mathbb{R}^D$ -valued centered Gaussian process with covariance

$$(1.9) \quad \mathbb{E}[w^q(\alpha)w^q(\alpha')^\top] = q(\alpha \wedge \alpha').$$

Its existence and properties are given in [22, Section 4]. Conditioned on $\mathfrak{R}$, let $(w_i^q)_{i \in \mathbb{N}}$ be i.i.d. copies of w^q and also independent of other randomness.

For $N \in \mathbb{N}$, $q \in \mathcal{Q}_\infty$, $t \geq 0$, the full energy function of the system will be chosen to be

$$(1.10) \quad \begin{aligned} H_{N;t,q}^G(\sigma, \alpha) = & NG(m_N) + \sqrt{2t}H_N(\sigma) - Nt\xi\left(\frac{\sigma\sigma^\top}{N}\right) \\ & + \sum_{i=1}^N w_i^q(\alpha) \cdot \sigma_{\bullet i} - \frac{1}{2}q(1) \cdot \sigma\sigma^\top. \end{aligned}$$

We are interested in the infinite-volume limit of the associated free energy, defined as

$$(1.11) \quad F_N^G(t, q) = -\frac{1}{N} \log \iint \exp(H_{N;t,q}^G(\sigma, \alpha)) dP_1^{\otimes N}(\sigma) d\mathfrak{R}(\alpha).$$

We also let $\overline{F}_N^G(t, q) = \mathbb{E}F_N^G(t, q) = \mathbb{E}_{H_N, (w_i^q)_{i \in \mathbb{N}}, (\chi_i)_{i \in \mathbb{N}}, \mathfrak{R}} F_N^G(t, q)$. In the following, if not specified, $\mathbb{E}$ always averages over the displayed randomness.

As is explained in full detail in Section 1.4, usual models such as (1.1) can be obtained as special cases of the Hamiltonian $H_{N;t,q}^G$. Briefly, to do so one can set $t = \beta^2/2$ and $q = 0$ and make a wise choice for G in order to absorb $-\frac{N\beta^2}{2}\xi\left(\frac{\sigma\sigma^\top}{N}\right)$ in the Mattis interaction.

Here, we have chosen to introduce the correction terms $Nt\xi(\sigma\sigma^\top/N)$ and $\frac{1}{2}q(1) \cdot \sigma\sigma^\top$ to simplify the forthcoming variational formulas. As explained in Section 1.4, they are easily removed. Also, notice that those terms are one-half of the variance of $\sqrt{2t}H_N(\sigma)$ and $\sum_{i=1}^N w_i^q(\alpha) \cdot \sigma_{\bullet i}$, respectively. So they correspond to drifts in exponential martingales.

The large deviation principle we prove holds almost surely with respect to the randomness of $(H_N)_{N \geq 1}$, $(\chi_i)_{i \in \mathbb{N}}$, and $\mathfrak{R}$. Therefore, throughout we will assume that those random variables are defined on a common probability space Ω to have a convenient way of stating those results.

Let $x \in \mathbb{R}^d$. When $G(m) = m \cdot x$, at $t = 0$ we have $\overline{F}_N^G(0, q) = \overline{F}_1^G(0, q)$ (see [22, Proposition 3.2]). In this case, we set $\psi(q; x) = \overline{F}_1^{m \mapsto m \cdot x}(0, q)$ which

can be expressed as

$$(1.12) \quad \psi(q; x) = -\mathbb{E} \log \iint \exp \left(\sqrt{2} w^q(\alpha) \cdot \tau - q(1) \cdot \tau \tau^\top + x \cdot h(\tau, \chi_1) \right) dP_1(\tau) d\mathfrak{A}(\alpha).$$

1.3. Main results. Throughout, given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we use $\|\cdot\|_{\text{Lip}(\mathcal{X}; \mathcal{Y})}$ to indicate the Lipschitz coefficient of functions from $\mathcal{X}$ to $\mathcal{Y}$. We let $\nabla \xi|_B$ be the restriction of $\nabla \xi$ to B where

$$(1.13) \quad B = \{a \in \mathbb{R}^{D \times D} : |a| \leq 1\}.$$

We let $t_\star \in (0, +\infty)$ be the positive real number defined by

$$(1.14) \quad t_\star^{-1} = \sup_{x' \in \mathbb{R}^d} \|\nabla_q \psi(\cdot; x')\|_{\text{Lip}(\mathcal{Q}_\infty; \mathcal{Q}_\infty)} \|\nabla \xi|_B\|_{\text{Lip}(B; \mathbb{R}^{D \times D})}.$$

Remark 1.3 (Lower bound for $t_\star$). Below, see (2.5) in Proposition 2.2, we justify that for every $x' \in \mathbb{R}^d$, the function $\psi(\cdot; x')$ is differentiable on $\mathcal{Q}_\infty$ and that its gradient $\nabla_q \psi(\cdot; x') : \mathcal{Q}_\infty \rightarrow \mathcal{Q}_\infty$ is a Lipschitz function with Lipschitz constant ≤ 16 . This yields the following lower bound

$$(1.15) \quad t_\star \geq \left(16 \|\nabla \xi|_B\|_{\text{Lip}(B; \mathbb{R}^{D \times D})} \right)^{-1} > 0.$$

We note that this bound is very loose and is unlikely to be optimal. $\square$

For every $x \in \mathbb{R}^d$, let $f = f(\cdot, \cdot; x) : \mathbb{R}_+ \times \mathcal{Q}_2 \rightarrow \mathbb{R}$ denote the unique solution⁴ of

$$(1.16) \quad \begin{cases} \partial_t f - \int \xi(\nabla f) = 0 \text{ on } (0, +\infty) \times \mathcal{Q}_2 \\ f(0, \cdot; x) = \psi(\cdot; x). \end{cases}$$

We refer the reader to the discussion at the beginning of Section 3 for more details on the precise meaning of (1.16) and the associated notion of viscosity solution.

Theorem 1.4 (Limit free energy). *Let $t < t_\star$ and $q \in \mathcal{Q}_\infty$. For every continuous function $G : \mathbb{R}^d \rightarrow \mathbb{R}$, we have*

$$(1.17) \quad \lim_{N \rightarrow +\infty} \overline{F}_N^G(t, q) = \inf_{m \in \mathbb{R}^d} \sup_{x \in \mathbb{R}^d} \{f(t, q; x) + m \cdot x - G(m)\}.$$

Moreover, for every $x \in \mathbb{R}^d$, there is a unique solution $p^x \in \mathcal{Q}_\infty$ of the fixed point equation

$$(1.18) \quad p = \nabla_q \psi(q + t \nabla \xi(p); x) \quad \text{for } p \in \mathcal{Q}_\infty,$$

and we have

$$(1.19) \quad f(t, q; x) = \psi(q + t \nabla \xi(p^x); x) - t \int_0^1 \theta(p^x(u)) du \quad \text{and} \quad \nabla_q f(t, q; x) = p^x.$$

⁴Below in Section 3 we give a precise meaning to this partial differential equation and explain in what sense it has a unique solution.

The convergence in (1.17) also happens almost surely due to the following.

Remark 1.5 (Concentration). The random variable $F_N^G(t, q)$ concentrates around its expectation $\mathbb{E}F_N^G(t, q)$ in the large N limit. Indeed, this can be deduced using the standard Gaussian concentration result (e.g. [47, Theorem 1.2]) for the Gaussian randomness in $F_N^G(t, q)$ together with the Efron–Stein inequality and the boundedness of the function $h(\cdot, \cdot)$ for random variables $(\chi_i)_i$. $\square$

The fixed point equation (1.18) can be understood as the critical point condition for the functional on the right-hand side of the first equation in (1.19). One can interpret the unique p^x solution of (1.18) as the Parisi measure, by identifying p^x with the law of $p^x(U)$ with U the uniform random variable over $[0, 1]$.

We also mention that representations similar to (1.19) have been obtained for the mutual information in the sparse stochastic block model (with no Mattis interaction) [28, Proposition 1.3].

Note that for every $x \in \mathbb{R}^d$, $t < t_\star$ and $q \in \mathcal{Q}_\infty$ we have

$$(1.20) \quad f(t, q; x) = \lim_{N \rightarrow +\infty} F_N^{m \mapsto x \cdot m}(t, q).$$

Remark 1.6 (Free energy of models without Mattis interaction). For models without Mattis interaction ($G = 0$), it is not known (but very strongly expected) that the free energy converges in the large N limit. Identifying the limiting value of the free energy is a major problem. In [46, 43] it was shown that for all $t \geq 0$ and $q \in \mathcal{Q}_\infty$,

$$(1.21) \quad \liminf_{N \rightarrow +\infty} F_N^0(t, q) \geq f(t, q; 0),$$

and it was conjectured that this lower bound is tight. As a consequence of (1.20) we deduce that this conjecture is at least verified for $t < t_\star$. $\square$

Definition 1.7 (Large deviation principle). Let $(X_N)_{N \geq 1}$ be a sequence of random variables in $\mathbb{R}^d$ with associated probability measures $(\mathbb{P}_N)_{N \geq 1}$ and let $I : \mathbb{R}^d \rightarrow [0, \infty]$. We say that the random variables X_N satisfy a large deviation principle under $\mathbb{P}_N$ with rate function I when for every Borel measurable set $A \subseteq \mathbb{R}^d$, we have

$$-\inf_{A^\circ} I \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N\{X_N \in A\} \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N\{X_N \in A\} \leq -\inf_{\bar{A}} I,$$

where A° and $\bar{A}$ are the interior and the closure of A , respectively.

The Gibbs measure associated to $H_{N;t,q}^G$ is the (random) probability measure defined on $\mathbb{R}^{D \times N} \times \text{supp}(\mathfrak{A})$ by

$$(1.22) \quad \langle \cdot \rangle_{N;t,q}^G \propto \exp(H_{N;t,q}^G(\sigma, \alpha)) dP_1^{\otimes N}(\sigma) \mathfrak{A}(\alpha).$$

For brevity, we also denote by $\langle \cdot \rangle_{N;t,q}^G$ its tensorization.

Recall the definition of $f(t, q, \cdot)$ in (1.20). We let $f^*(t, q; \cdot)$ denote the convex dual of $-f(t, q; \cdot)$, namely,

$$(1.23) \quad f^*(t, q; m) = \sup_{x \in \mathbb{R}^d} \{x \cdot m + f(t, q; x)\}, \quad \forall m \in \mathbb{R}^d.$$

Recall that the random variables $(H_{N;t,q}^G)_{N \geq 1}$ and $(\chi_i)_{i \in \mathbb{N}}$ are defined on a common probability space Ω . For a random variable $X : \Omega \rightarrow \mathfrak{X}$, we let $X^\omega \in \mathfrak{X}$ denote the realization of the random variable X on the event $\omega \in \Omega$. Recall the mean magnetization m_N defined in (1.8). Our next main result is a (quenched) large deviation principle for the random variables m_N under $\langle \cdot \rangle_{N;t,q}^G$.

Theorem 1.8 (Large deviations of m_N). *Let $t < t_*$ and $q \in \mathcal{Q}_\infty$. There is a full-measure subset $\Omega' \subseteq \Omega$ such that the following holds. For every $\omega \in \Omega'$ and every continuous function $G : \mathbb{R}^d \rightarrow \mathbb{R}$, the random variable m_N satisfies a large deviation principle under $\langle \cdot \rangle_{N;t,q}^{G,\omega}$ with rate function $I_{t,q}^G$ defined by*

$$(1.24) \quad I_{t,q}^G(m) = -G(m) + f^*(t, q; m) + \sup_{m'} \{G(m') - f^*(t, q; m')\}, \quad \forall m \in \mathbb{R}^d.$$

For $y \in S^D$, we define

$$(1.25) \quad \hat{y}(s) : [0, 1) \rightarrow S^D$$

to be the constant path with value y . For $\kappa \in L^1$, we define

$$(1.26) \quad \check{\kappa} = (\kappa)^\vee = \int_0^1 \kappa(r) dr \in S^D.$$

When the path q is chosen to be $\hat{y}$ and particularly $\hat{0}$, we show that the representation for the free energy derived in Theorem 1.4 reduces to a replica symmetric formula (Theorem 1.9). This is not surprising and corresponds to the traditional picture of replica symmetry at low t as previously mentioned.

For every $y \in S_+^D$ and $x \in \mathbb{R}^d$, we set

$$(1.27) \quad \phi(y; x) = -\mathbb{E} \log \int \exp \left(\sqrt{2y} \eta \cdot \tau - y \cdot \tau \tau^\top + x \cdot h(\tau, \chi_1) \right) dP_1(\tau),$$

where η is an independent standard $\mathbb{R}^D$ -valued Gaussian vector (i.e. $\mathbb{E} \eta \eta^\top$ is the $D \times D$ -identity matrix) and $\mathbb{E}$ averages over both η and χ_1 . We denote the derivative in the first variable of $\phi(\cdot; x)$ as $\nabla_y \phi(\cdot; x)$.

Theorem 1.9 (Replica symmetric form). *Let $x \in \mathbb{R}^d$. Define $g(\cdot, \cdot; x) : [0, t_*) \times S_+^D \rightarrow \mathbb{R}$ by*

$$(1.28) \quad g(t, y; x) = f(t, \hat{y}; x).$$

The function $g(\cdot, \cdot; x)$ is differentiable on $[0, t_) \times S_+^D$ and satisfies*

$$(1.29) \quad \begin{cases} \partial_t g(t, y; x) - \xi(\nabla_y g(t, y; x)) = 0 & \text{for } (t, y) \in (0, t_*) \times S_+^D, \\ g(0, y; x) = \phi(y; x) & \text{for } y \in S_+^D. \end{cases}$$

Moreover, at every $(t, y) \in (0, t_*) \times S_+^D$, there is a unique solution $z \in S_+^D$ to the fixed point equation

$$(1.30) \quad z' = \nabla_y \phi(t \nabla \xi(z'); x) \quad \text{for } z' \in S_+^D,$$

and we have

$$(1.31) \quad g(t, y; x) = \phi(y + t \nabla \xi(z); x) - t \theta(z) \quad \text{and} \quad \nabla_y g(t, y; x) = z.$$

As before, the fixed point equation (1.30) can be understood as the critical point condition for the right-hand side of the first equation in (1.31).

Remark 1.10 (Gaussian external field). When $q = \hat{y}$, we can interpret the model as a non-enriched model with a Gaussian external field. Recall the covariance structure of $w^q(\alpha)$ from (1.9). When $q = \hat{y}$ for some $y \in S_+^D$, we have

$$(1.32) \quad w^{\hat{y}}(\alpha) \stackrel{d}{=} \sqrt{y} \eta,$$

where η is an independent standard $\mathbb{R}^D$ -valued Gaussian vector. Therefore, inside $H_{N;t,q}^G(\sigma, \alpha)$, we have

$$(1.33) \quad \sum_{i=1}^N w_i^{\hat{y}}(\alpha) \cdot \sigma_{\bullet i} \stackrel{d}{=} \sqrt{y} \sum_{i=1}^N \eta_i \cdot \sigma_{\bullet i},$$

where (η_i) consists of i.i.d. standard $\mathbb{R}^D$ -valued Gaussian vectors. Since this quantity no longer depends on α , we can rewrite $H_{N;t,\hat{y}}^G(\sigma, \alpha)$ simply as $H_{N;t,y}^G(\sigma)$, where y determines the Gaussian external field $\sqrt{y} \sum_{i=1}^N \eta_i \cdot \sigma_{\bullet i}$. In view of (1.12) and (1.27), we can use (1.32) to see that for every $y \in S_+^D$ and $x \in \mathbb{R}^d$,

$$(1.34) \quad \phi(y; x) = \psi(\hat{y}; x). \quad \square$$

Remark 1.11 (Replica symmetric formula for the limit free energy). At $q = \hat{y}$ for $y \in S_+^D$ which includes the non-enriched case $q = \hat{0}$, we can thus replace $f(t, q; x)$ appearing in Theorems 1.4 and 1.8 by the replica-symmetric counterpart $g(t, y; x)$. Recall that we interpret p^x appearing in Theorem 1.4 as the Parisi measure. Then, in this case, p^x corresponds to $\widehat{z^x}$, where z^x is the unique solution of (1.30) at x . Hence, indeed the system is replica-symmetric. $\square$

When q is not a constant path, then p^x will not be constant. Here, the symmetry breaking is not intrinsic to system but rather caused by q .

Remark 1.12 (Replica symmetry breaking induced by q). Fix $x \in \mathbb{R}^d$. Let $(t, q) \in [0, t_*) \times \mathcal{Q}_2$ and assume that q as a path has a jump at $s \in (0, 1)$. The map $\nabla_q \psi(\cdot; x)$ preserves jumps (see Lemma 4.7). Therefore, the unique solution $p^x \in \mathcal{Q}_\infty$ in Theorem 1.4 has also a jump at s . In particular, if q is strictly increasing on $[0, 1)$, then so is p^x . $\square$

Our main results are stated for models based on vector spin glasses. The same conclusions can be deduced for the following models as well.

Remark 1.13 (Multi-type and multi-species models). Further simplification can be made if ξ only depends on the diagonal of the matrix. We call such a model multi-type. In this setting, only the diagonal in the self-overlap matrix matters, and thus the free energy can be encoded with paths taking values in $\mathbb{R}_+^D$ rather than S_+^D . A multi-type model is a special case of multi-species models. In Appendix A, we describe how our main results can be adapted to multi-species models. We note that passing back and forth between vector models and multi-species models is largely a technical matter and does not introduce significant new challenges. Essentially all the results mentioned so far on the free energy of vector spin glasses remain valid in this setting [9, 48]. $\square$

1.4. Recovering concrete models from the general setting. The main results of this paper are stated for very general models as introduced in Section 1.2. Let us explain how to recover results about more usual models, like (1.1), from those.

Given $h : \mathbb{R}^D \times \mathbb{R}^L \rightarrow \mathbb{R}^d$ in (1.11), we define $\tilde{h} : \mathbb{R}^D \times \mathbb{R}^L \rightarrow \mathbb{R}^d \times S^D$ through

$$(1.35) \quad \tilde{h}(\tau, \chi) = (h(\tau, \chi), \tau\tau^\top).$$

Henceforth, we isometrically identify $\mathbb{R}^d \times S^D$ with $\mathbb{R}^{\tilde{d}}$ for some $\tilde{d} \in \mathbb{N}$. Given $G : \mathbb{R}^d \rightarrow \mathbb{R}$ in (1.11), we introduce $\tilde{G} : \mathbb{R}^d \times S^D \rightarrow \mathbb{R}$ given by

$$(1.36) \quad \tilde{G}(m, r) = G(m) + \frac{\beta^2}{2} \xi(r).$$

Then, we consider $H_{N;t,q}^G$ in (1.10) with G, h, d , therein substituted with $\tilde{G}, \tilde{h}, \tilde{d}$ introduced above and with $t = \beta^2/2$ and $q = \hat{0}$ (constant path taking the value 0), we denote the resulting Hamiltonian $\tilde{H}_N$. Observe that we have

$$(1.37) \quad \tilde{H}_N(\sigma) = \beta H_N(\sigma) + N \tilde{G} \left(\frac{1}{N} \sum_{i=1}^N \tilde{h}(\sigma_i, \chi_i) \right) - \frac{N\beta^2}{2} \xi \left(\frac{\sigma\sigma^\top}{N} \right)$$

$$(1.38) \quad = \beta H_N(\sigma) + N G \left(\frac{1}{N} \sum_{i=1}^N h(\sigma_{\bullet i}, \chi_i) \right).$$

In particular, observe that the model described in (1.1) is recovered by choosing $D = 2$, $d = 2$, $L = 2$, $h(\sigma, \chi) = (\text{sgn}(\sigma_1 \chi_1), \text{sgn}(\sigma_2 \chi_2))$, $G(m) = m_1 + m_2$, $\xi(a) = a_{11}a_{22}$, and P_1 is the uniform measure on $\{-1, 1\}^2$.

Remark 1.14 (About Theorem 1.1). Observe that thanks to the specialization of the general models from Section 1.2 to (1.1) given in this section, Theorem 1.1 can (almost) be recovered from Theorem 1.4, Theorem 1.8 and Theorem 1.9. The small difference being that in those results the paths in $\mathcal{Q}$ are S_+^2 valued, while in Theorem 1.1 the expression for the limit free energy comes from constant $\mathbb{R}_+^2$ valued paths. This difference originates from the fact that the model described in (1.1) is a multi-type model in the sense defined in Remark 1.13. Indeed, $\xi(a) = a_{11}a_{22}$ does not depend on the whole input matrix but only on its diagonal elements. To recover Theorem 1.1 as

stated, one can use the version of the main results stated in Appendix A instead. The lower bound $\beta_\star \geq \frac{1}{2\sqrt{2}}$ comes from setting $\beta_\star = \sqrt{2t_\star}$ and the lower bound (1.15) for $t_\star$. $\square$

1.5. Organization of the paper. In Section 2, we establish some regularities of the function ψ that are useful later on; these results are gathered in Proposition 2.2. In Section 3, we show that the solution of (1.16) is differentiable on the time interval $[0, t_\star)$. Finally, in Section 4, we prove our main results. We start with a proof of (1.20), this is the key aspect of our argument, and it relies on the content of all the previous sections, see Corollary 4.3. We also crucially verify, thanks to (1.20), that $x \mapsto \lim_{N \rightarrow +\infty} F_N^{x \mapsto x^m}(t, q)$ is continuously differentiable in x , see Lemma 4.4. Once those two key results are proven, we can deduce Theorem 1.4 and Theorem 1.8 using classical results from large deviation theory.

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2. REGULARITY OF ψ

Definition 2.1 (Fréchet differentiability). Let $(E, |\cdot|_E)$ be a Banach space and denote by $\langle \cdot, \cdot \rangle_E$ the canonical pairing between E and E^* . Let G be a subset of E and let $q \in G$. A function $g : G \rightarrow \mathbb{R}$ is **Fréchet differentiable** at $q \in G$ if there exists a unique $y^* \in E^*$ such that the following holds:

$$\lim_{\substack{q' \rightarrow q \\ q' \in G}} \frac{g(q') - g(q) - \langle y^*, q' - q \rangle_E}{|q' - q|_E} = 0.$$

In such a case, we call y^* the **Fréchet derivative** of g at q , and we denote it by $\nabla g(q)$ or $\nabla_q g(q)$.

More generally, let $(E', |\cdot|_{E'})$ be another Banach space. Let G, G' be subsets of E and E' , respectively, and let $q \in G$. A function $g : G \rightarrow G'$ is **Fréchet differentiable** at $q \in G$ if there exists a unique bounded operator $A : E \rightarrow E'$ such that

$$\lim_{\substack{q' \rightarrow q \\ q' \in G}} \frac{|g(q') - g(q) - A(q' - q)|_{E'}}{|q' - q|_E} = 0.$$

Again, in such a case, we denote A as $\nabla g(q)$ or $\nabla_q g(q)$.

We will use the operator norm on the set of bounded operators $A : E \rightarrow E'$, defined as

$$(2.1) \quad \|A\|_{\text{Op}(E, E')} = \sup_{q \in E \setminus \{0\}} \frac{|Aq|_{E'}}{|q|_E}.$$

Proposition 2.2 (Regularity of ψ). *The following holds, where we write $L^p = L^p([0, 1]; S^D)$ for $p \in [1, \infty]$.*

- (1) *The function ψ can be extended to $\mathcal{Q}_1 \times \mathbb{R}^d$ and satisfies, for any $q, q' \in \mathcal{Q}_1$ and $x, x' \in \mathbb{R}^d$,*

$$(2.2) \quad |\psi(q; x) - \psi(q'; x')| \leq |q - q'|_{L^1} + |h|_{L^\infty} |x - x'|.$$

- (2) *For each $x \in \mathbb{R}^d$, the restriction of the function $\psi(\cdot; x)$ to $\mathcal{Q}_2$ is Fréchet differentiable everywhere. The derivative $\nabla_q \psi(\cdot; x)$ can be extended to $\mathcal{Q}_1$ and satisfies, for any $q, q' \in \mathcal{Q}_1$, $x, x' \in \mathbb{R}^d$, and $r \in [1, +\infty]$,*

$$(2.3) \quad \nabla_q \psi(q; x) \in \mathcal{Q}, \quad |\nabla_q \psi(q; x)|_{L^\infty} \leq 1,$$

$$(2.4) \quad |\psi(q'; x) - \psi(q; x) - \langle q' - q, \nabla_q \psi(q; x) \rangle_{L^2}| \leq 8|q - q'|_{L^2}^2,$$

$$(2.5) \quad |\nabla_q \psi(q; x) - \nabla_q \psi(q'; x')|_{L^r} \leq 16|q - q'|_{L^r} + 4|h|_{L^\infty} |x - x'|.$$

- (3) *For each $q \in \mathcal{Q}_1$, the function $\psi(q; \cdot)$ is differentiable everywhere on $\mathbb{R}^d$ and its derivative satisfies, for $q, q' \in \mathcal{Q}_1$, $x, x' \in \mathbb{R}^d$,*

$$(2.6) \quad |\nabla_x \psi(q; x)| \leq |h|_{L^\infty},$$

$$(2.7) \quad |\psi(q; x') - \psi(q; x) - (x' - x) \cdot \nabla_x \psi(q; x)| \leq |h|_{L^\infty}^2 |x - x'|^2,$$

$$(2.8) \quad |\nabla_x \psi(q; x) - \nabla_x \psi(q'; x')| \leq 8|h|_{L^\infty} |q - q'|_{L^1} + 2|h|_{L^\infty}^2 |x - x'|.$$

- (4) *For each $x \in \mathbb{R}^d$, the function $\nabla_q \psi(\cdot; x) : \mathcal{Q}_{2s} \rightarrow L^s$ is Fréchet differentiable everywhere for every $s \in [1, +\infty]$ and satisfies, for $q, q' \in \mathcal{Q}_{2s}$,*

$$(2.9) \quad |\nabla_q \psi(q'; x) - \nabla_q \psi(q; x) - \nabla_q^2 \psi(q; x)(q' - q)|_{L^s} \leq C|q - q'|_{L^{2s}}^2/2,$$

for some absolute constant $C > 0$ independent of x and s , where $\nabla_q^2 \psi(q; x)$ is viewed as a bounded linear operator from L^{2s} to L^s (but actually enjoys better regularity as in (2.10)). The derivative $\nabla_q^2 \psi(\cdot; x)$ can be extended to $\mathcal{Q}_1$ and satisfies, for every $q, q' \in \mathcal{Q}_1$ and every $p, q \in [1, +\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} \leq 1$,

$$(2.10) \quad \|\nabla_q^2 \psi(q; x)\|_{\text{Op}(L^p; L^p)} \leq 16,$$

$$(2.11) \quad \|\nabla_q^2 \psi(q; x) - \nabla_q^2 \psi(q'; x)\|_{\text{Op}(L^p; L^{\frac{pq}{p+q}})} \leq C|q - q'|_{L^q},$$

for the same constant C in (2.9).

One useful case of (2.11) is when $p = q = 2s$ for some $s \in [1, +\infty]$ and thus $\frac{pq}{p+q} = s$.

Proof of Proposition 2.2. The argument is based on computing directional derivatives in q and x . The computation is straightforward for the latter, and we explain the former. Fix any $q, q' \in \mathcal{Q}_s$ and let $\lambda \in [0, 1]$. We want to compute the derivative in λ along $\lambda \mapsto q + \lambda(q' - q)$. The key observation is that $w^{q+\lambda(q'-q)}(\alpha)$ inside $\psi(q + \lambda(q' - q))$ (see (1.12)) has the same law as

$\sqrt{1-\lambda}w^q(\alpha) + \sqrt{\lambda}w^{q'}(\alpha)$ where $w^q(\alpha)$ and $w^{q'}(\alpha)$ are independent. Hence, we can carry out the computation in terms of $\sqrt{1-\lambda}w^q(\alpha) + \sqrt{\lambda}w^{q'}(\alpha)$. The recipe is as follows: first, we differentiate a quantity in λ to get an expression involving $w^q(\alpha)$ and $w^{q'}(\alpha)$, and then we apply the Gaussian integration by parts [29, Theorem 4.6] to them to get the final expression. Since these procedures are standard, we omit the intermediate steps and refer to the computations in the proof of [22, Proposition 5.1] as an example.

Throughout the proof, we use the following notation. For $\kappa \in L^\infty([0, 1]; S^D)$ and $\ell, \ell' \in \mathbb{N}$, we write

$$R_\kappa^{\ell, \ell'} = \kappa \left(\alpha^\ell \wedge \alpha^{\ell'} \right) \cdot \tau^\ell (\tau^{\ell'})^\top.$$

where $(\tau^\ell)_{\ell \in \mathbb{N}}$ are independent samples from some Gibbs measure that will be clear from the context. For each $a \in \mathbb{R}^d$ and $\ell \in \mathbb{N}$, we write

$$h^\ell = h(\tau^\ell, \chi_1) \quad \text{and} \quad h_a^\ell = a \cdot h(\tau^\ell, \chi_1).$$

In the following, let $\lambda \in [0, 1]$. Recall the definition of the Gibbs measure $\langle \cdot \rangle_{N; t, q}^G$ in (1.22). For $q, q' \in \mathcal{Q}_\infty$ and $x, x' \in \mathbb{R}^d$, we write

$$\langle \cdot \rangle_\lambda = \langle \cdot \rangle_{1; 0, q + \lambda(q' - q)}^{m \mapsto m \cdot (x + \lambda(x' - x))},$$

where it is also possible that we take $q = q'$ or $x = x'$, which will be clear from the context.

Since $\alpha^\ell \wedge \alpha^{\ell'}$ distributes uniformly over $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_\lambda$ for $\ell \neq \ell'$ and $|\tau \tau^\top| \leq 1$ due to the support of τ on the unit ball in $\mathbb{R}^D$, we often use the the following estimates:

$$(2.12) \quad \mathbb{E} \left| R_\kappa^{\ell, \ell'} \right|_\lambda \leq |\kappa|_{L^1} \quad \text{and} \quad \mathbb{E} \left| R_\kappa^{\ell, \ell'} R_\eta^{\ell'', \ell'''} \right|_\lambda \leq |\kappa|_{L^p} |\eta|_{L^q},$$

where $\ell \neq \ell'$, $\ell'' \neq \ell'''$, $\kappa, \eta \in L^\infty([0, 1]; S^D)$, and $\frac{1}{p} + \frac{1}{q} = 1$. The last inequality follows from the Hölder's inequality applied to the measure $\mathbb{E} \langle \cdot \rangle_\lambda$. Later, we will also use the generalization of the last one to three components.

We also mention that the correction term $-q(1) \cdot \tau \tau^\top$ in (1.12) becomes $-(q + \lambda(q' - q))(1) \cdot \tau \tau^\top$ when we compute the direction derivative. The presence of this term will cancel all $R_\kappa^{\ell, \ell'}$ and thus in the final expression we only have terms $R_\kappa^{\ell, \ell'}$ with $\ell \neq \ell'$. This is an important reason for the usefulness of the correction.

Lastly, slightly abusing the notation, we denote by $\langle \cdot, \cdot \rangle_{L^2}$ not only the inner product in $L^2([0, 1]; S^D)$ but also the natural pairing in $L^p([0, 1]; S^D)$ and its dual. We also use the shorthand $L^p = L^p([0, 1]; S^D)$. The only exception is that L^∞ in $|h|_{L^\infty}$ means $L^\infty(\mathbb{R}^D \times \mathbb{R}^L; \mathbb{R}^d)$.

With the above setup, we are ready to prove the lemma part by part.

Part (1). Recall that ψ is initially defined on $\mathcal{Q}_\infty \times \mathbb{R}^d$ as in (1.12). For $q, q' \in \mathcal{Q}_\infty$ and $x, x' \in \mathbb{R}^d$, we can compute

$$(2.13) \quad \frac{d}{d\lambda} \psi(q + \lambda(q' - q); x + \lambda(x' - x)) = \mathbb{E} \left\langle R_{q'-q}^{1,2} \right\rangle_\lambda - \mathbb{E} \langle h_{x'-x}^1 \rangle_\lambda.$$

Using (2.12), the last term is bounded by $|q' - q|_{L^1} + |h|_{L^\infty}$. Therefore, we get that (2.2) holds for $q, q' \in \mathcal{Q}_\infty$ and $x, x' \in \mathbb{R}^d$. By this continuity, we can extend the definition of ψ to $\mathcal{Q}_1 \times \mathbb{R}^d$ and accordingly extend (2.2) as desired.

Part (2). For each $q \in \mathcal{Q}_\infty$ and $x \in \mathbb{R}^d$, we define $\nabla_q \psi(q; x) \in L^2$ as the element determined through

$$(2.14) \quad \langle \kappa, \nabla_q \psi(q; x) \rangle_{L^2} = \mathbb{E} \langle R_\kappa^{1,2} \rangle_0, \quad \forall \kappa \in L^\infty.$$

Indeed, setting $x = x'$ in (2.13), we get

$$(2.15) \quad \frac{d}{d\lambda} \psi(q + \lambda(q' - q); x) = \langle q' - q, \nabla_q \psi(q; x) \rangle_{L^2}, \quad \forall q, q' \in \mathcal{Q}_\infty, x \in \mathbb{R}^d.$$

This suggests that $\nabla_q \psi(q; x)$ is the correct derivative, which we will justify later after gathering enough properties.

Since $\langle R_\kappa^{1,2} \rangle_\lambda \leq |\kappa|_{L^1}$ due to (2.12) and the density of L^∞ in L^2 , we get from (2.14) that

$$(2.16) \quad |\nabla_q \psi(q; x)|_{L^\infty} \leq 1, \quad \forall q \in \mathcal{Q}_\infty, x \in \mathbb{R}^d.$$

For $\kappa \in L^\infty$, $q, q' \in \mathcal{Q}_\infty$, and $x \in \mathbb{R}^d$, we can compute

$$(2.17) \quad \frac{d}{d\lambda} \langle \kappa, \nabla_q \psi(q + \lambda(q' - q); x) \rangle_{L^2} = 2\mathbb{E} \left\langle R_\kappa^{1,2} \left(R_{q'-q}^{1,2} - 4R_{q'-q}^{1,3} + 3R_{q'-q}^{3,4} \right) \right\rangle_\lambda.$$

By (2.12), the right-hand side is bounded by $16|\kappa|_{L^{r^*}} |q - q'|_{L^r}$ with $\frac{1}{r^*} + \frac{1}{r} = 1$. Integrating in λ and using the density of L^∞ in L^{r^*} , we get

$$(2.18) \quad |\nabla_q \psi(q; x) - \nabla_{q'} \psi(q'; x)|_{L^r} \leq 16|q - q'|_{L^r}, \quad \forall q, q' \in \mathcal{Q}_\infty, x \in \mathbb{R}^d.$$

Setting $r = 1$ and using (2.16), we extend $\nabla \psi(q, x)$ by continuity to $q \in \mathcal{Q}_1$ and achieve the L^∞ -bound in (2.3). First property in (2.3) is proved in [22, Corollary 5.2].

Setting $\kappa = q' - q$ in (2.17) and $r = 2$ in the bound mentioned below (2.17) and comparing with (2.15), we get $\left| \frac{d^2}{d\lambda^2} \psi(q + \lambda(q' - q); x) \right| \leq 16|q - q'|_{L^2}$. This, along with (2.15) and Taylor's expansion, gives that for any fixed $x \in \mathbb{R}^d$,

$$(2.19) \quad |\psi(q'; x) - \psi(q; x) - \langle q' - q, \nabla_q \psi(q; x) \rangle_{L^2}| \leq 8|q - q'|_{L^2}^2, \quad \forall q, q' \in \mathcal{Q}_\infty.$$

Since we have shown that $\psi(\cdot; x)$ and $\nabla_q \psi(\cdot; x)$ can be extended continuously to $\mathcal{Q}_1$ and that the latter is also bounded, we can now extend (2.19) to $q, q' \in \mathcal{Q}_2$. This proves that $\psi(\cdot; x)$ is Fréchet differentiable everywhere on $\mathcal{Q}_2$ satisfying (2.4) and that $\nabla_q \psi(q; x)$ defined via (2.14) is its derivative.

To complete this part, it remains to show (2.5). For $\kappa \in L^\infty$, $q \in \mathcal{Q}_\infty$ and $x, x' \in \mathbb{R}^d$, we compute from (2.14) that

$$\frac{d}{d\lambda} \langle \kappa, \nabla_q \psi(q; x + \lambda(x' - x)) \rangle_{L^2} = \mathbb{E} \langle R_\kappa^{1,2} (h_{x'-x}^1 + h_{x'-x}^1 - 2h_{x'-x}^1) \rangle_\lambda.$$

Using (2.12) to bound the right-hand side by $4|h|_{L^\infty}|\kappa|_{L^1}|x-x'|$ and using the density of L^∞ in L^1 , we get

$$|\nabla_q \psi(q; x) - \nabla_q \psi(q; x')| \leq 4|\kappa|_{L^1}|x - x'|, \quad \forall q \in \mathcal{Q}_\infty, x, x' \in \mathbb{R}^d.$$

As before, we can extend this relation to $q \in \mathcal{Q}_1$, which along with (2.18) extended to $\mathcal{Q}_1$ yields (2.5).

Part (3). We can easily compute

$$\nabla_x \psi(q; x) = -\mathbb{E} \langle h^1 \rangle_0, \quad |\nabla_x \psi(q; x)| \leq |h|_{L^\infty}, \quad \forall q \in \mathcal{Q}_\infty, x \in \mathbb{R}^d.$$

For $q, q' \in \mathcal{Q}_\infty$ and $x \in \mathbb{R}^d$, we can then compute

$$\frac{d}{d\lambda} \nabla_x \psi(q + \lambda(q' - q); x) = 4\mathbb{E} \left\langle h^1 \left(R_{q'-q}^{1,2} - R_{q'-q}^{2,3} \right) \right\rangle_\lambda.$$

Using (2.12) to bound the right-hand side, we get

$$(2.20) \quad |\nabla_x \psi(q; x) - \nabla_x \psi(q'; x)| \leq 8|h|_{L^\infty}|q - q'|_{L^1} \quad \forall q \in \mathcal{Q}_\infty, x \in \mathbb{R}^d.$$

Therefore, we can extend $\nabla_x \psi(q; x)$ by continuity to $q \in \mathcal{Q}_1$ and get the bound in (2.6). Notice that we have yet to verify that the extended $\nabla_x \psi(q; x)$ is indeed the derivative at $q \in \mathcal{Q}_1$.

To do so, we start by computing, for $q \in \mathcal{Q}_\infty$ and $x, x' \in \mathbb{R}^d$,

$$(2.21) \quad \frac{d}{d\lambda} \nabla_x \psi(q; x + \lambda(x' - x)) = -\mathbb{E} \langle h^1 (h_{x'-x}^1 - h_{x'-x}^2) \rangle_\lambda,$$

which implies

$$|\nabla_x \psi(q; x) - \nabla_x \psi(q; x')| \leq 2|h|_{L^\infty}^2|x - x'|, \quad \forall q \in \mathcal{Q}_\infty, x, x' \in \mathbb{R}^d.$$

Again, (2.20) allows us to extend this bound to $q \in \mathcal{Q}_1$, which along with the extension of (2.20) to $\mathcal{Q}_1$ yields (2.8).

Multiply both sides in (2.21) by $x' - x$, we can get $|\frac{d}{d\lambda} \psi(q; x + \lambda(x' - x))| \leq 2|h|_{L^\infty}^2|x - x'|^2$, which by Taylor's expansion gives that, uniformly in $x, x' \in \mathbb{R}^d$,

$$|\psi(q; x') - \psi(q; x) - (x' - x) \cdot \nabla_x \psi(q; x)| \leq |h|_{L^\infty}^2|x - x'|^2, \quad \forall q \in \mathcal{Q}_\infty.$$

As before, we can extend this by continuity to $q \in \mathcal{Q}_1$. This verifies that $\nabla \psi(q; \cdot)$ is differentiable at $q \in \mathcal{Q}_1$, (2.7), and $\nabla_x \psi(q; x)$ is indeed the derivative.

Part (4). We fix any $x \in \mathbb{R}^d$ and write $\psi = \psi(\cdot; x)$. For any $\mathbf{p} \in [0, +\infty]$ and $q \in \mathcal{Q}_\infty$, we define $\nabla_q^2 \psi(q)$ as a linear operator from $L^{\mathbf{p}}$ to $L^{\mathbf{p}}$ by setting

$$(2.22) \quad \langle \kappa, \nabla_q^2 \psi(q) \eta \rangle_{L^2} = 2\mathbb{E} \langle R_\kappa^{1,2} (R_\eta^{1,2} - 4R_\eta^{1,3} + 3R_\eta^{3,4}) \rangle_0, \quad \forall \kappa, \eta \in L^\infty.$$

Notice that this definition can be made simultaneously for all $\mathbf{p}$. Using (2.12), we can bound the right-hand side by $16|\kappa|_{L^{\mathbf{p}^*}}|\eta|_{L^{\mathbf{p}}}$ with $\frac{1}{\mathbf{p}^*} + \frac{1}{\mathbf{p}} = 1$, which implies

$$(2.23) \quad \|\nabla_q^2 \psi(q)\|_{\text{Op}(L^{\mathbf{p}}; L^{\mathbf{p}})} \leq 16, \quad \forall q \in \mathcal{Q}_\infty.$$

Comparing with (2.17), we get, for $q, q' \in \mathcal{Q}_\infty$ and $\kappa \in L^\infty$,

$$(2.24) \quad \frac{d}{d\lambda} \langle \kappa, \nabla_q \psi(q + \lambda(q' - q)) \rangle_{L^2} = \langle \kappa, \nabla_q^2 \psi(q + \lambda(q' - q))(q' - q) \rangle_{L^2}.$$

We will extend the definition of $\nabla_q^2 \psi$ to $\mathcal{Q}_1$ and verify that it is indeed a Fréchet derivative. But first, we need to derive some estimates. For $\kappa, \eta \in L^\infty$ and $q, q' \in \mathcal{Q}_\infty$, we can compute from (2.22) that

$$(2.25) \quad \frac{d}{d\lambda} \langle \kappa, \nabla_q^2 \psi(q + \lambda(q' - q)) \eta \rangle_{L^2} = \mathbb{E} \langle R_\kappa^{1,2} (R_\eta^{1,2} - 4R_\eta^{1,3} + 3R_\eta^{3,4}) \mathcal{P}(R_{q'-q}) \rangle_\lambda,$$

where $\mathcal{P}$ is a homogeneous polynomial of degree one and depends on finitely many entries in $R_{q'-q} = (R_{q'-q}^{\ell, \ell'})_{\ell \neq \ell'}$. Using an extension of the estimate in (2.12), we can bound the right-hand side in (2.25) by $C|\kappa|_{L^r}|\eta|_{L^p}|q - q'|_{L^q}$ for some absolute constant $C > 0$, where $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1$. Notice that the conjugate of r is $r^* = \frac{pq}{p+q}$. From this, we can get

$$(2.26) \quad \|\nabla_q^2 \psi(q) - \nabla_q^2 \psi(q')\|_{\text{Op}(L^p; L^{\frac{pq}{p+q}})} \leq C|q - q'|_{L^q}, \quad \forall q, q' \in \mathcal{Q}_\infty.$$

In this relation, setting $r = p = \infty$ and $q = 1$, we can thus extend by continuity the definition of $\nabla_q^2 \psi(q)$ to every $q \in \mathcal{Q}_1$ as a bounded linear operator from L^∞ to L^1 . Then, we want to accordingly extend (2.23). Let $q \in \mathcal{Q}_1$ and let $(q_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{Q}_\infty$ convergent to q in L^1 . Then, for any $\kappa, \eta \in L^\infty$, we have

$$\langle \kappa, \nabla_q^2 \psi(q) \eta \rangle_{L^2} \leq \langle \kappa, \nabla_q^2 \psi(q_n) \eta \rangle_{L^2} + \langle \kappa, (\nabla_q^2 \psi(q) - \nabla_q^2 \psi(q_n)) \eta \rangle_{L^2}.$$

By (2.26) with $r = p = \infty$ and $q = 1$, the last term vanishes as $n \rightarrow \infty$. Taking this limit and using (2.23) for q_n , we can obtain from the above display that (2.23) holds for $q \in \mathcal{Q}_1$, which gives (2.10) as desired. Therefore, for every $q \in \mathcal{Q}_1$, $\nabla_q^2 \psi(q)$ is also a bounded operator from $L^p \rightarrow L^p$. Due to $p \geq \frac{pq}{p+q}$, it is also an operator from $L^p \rightarrow L^{\frac{pq}{p+q}}$. Therefore, we can extend (2.26) by continuity to $q, q' \in \mathcal{Q}_1$, which is exactly (2.11).

Lastly, we verify that $\nabla_q^2 \psi(q)$ indeed defines a Fréchet derivative. Setting $\eta = q' - q$ in (2.25), comparing it with (2.24), and using the estimate mentioned below (2.25) with $r = \frac{s}{s-1}$ and $p = q = 2s$, we get, for $\kappa \in L^\infty$ and $q, q' \in \mathcal{Q}_\infty$,

$$\left| \frac{d^2}{d\lambda^2} \langle \kappa, \nabla_q \psi(q + \lambda(q' - q)) \rangle_{L^2} \right| \leq C|\kappa|_{L^{\frac{s}{s-1}}} |q - q'|_{L^{2s}}^2.$$

Using this, Taylor's expansion, and (2.24), we get

$$\left| \langle \kappa, \nabla_q \psi(q') - \nabla_q \psi(q) - \nabla_q^2 \psi(q)(q' - q) \rangle_{L^2} \right| \leq C|\kappa|_{L^{\frac{s}{s-1}}} |q - q'|_{L^{2s}}^2/2.$$

and thus

$$\|\nabla_q \psi(q') - \nabla_q \psi(q) - \nabla_q^2 \psi(q)(q' - q)\|_{L^s} \leq C|q - q'|_{L^{2s}}^2/2, \quad \forall q, q' \in \mathcal{Q}_\infty.$$

We can extend this to $q, q' \in \mathcal{Q}_{2s}$ by using (2.5) (with $r = s$), (2.10) (with $p = s$), and (2.11) (with $p = q = 2s$). This implies the statement on the Fréchet differentiability of this part and (2.9), completing the proof. $\square$

We also record a simple result based on Proposition 2.2. Define the set

$$(2.27) \quad \mathcal{Q}_{\infty, \leq 1} = \{p \in \mathcal{Q}_{\infty} : |p|_{L^{\infty}} \leq 1\}.$$

For every $t \geq 0$, $p, q \in \mathcal{Q}_{\infty}$, and $x \in \mathbb{R}^d$, we define

$$(2.28) \quad \mathcal{P}_{t,q}(p; x) = \psi(q + t\nabla\xi(p); x) - t \int_0^1 \theta(p(u)) du.$$

Lemma 2.3. *Let $t \in \mathbb{R}_+$ and $q \in \mathcal{Q}_2$. There is a constant $C > 0$ such that, for every $x \in \mathbb{R}^d$, if $p^x \in \mathcal{Q}_{\infty, \leq 1}$ satisfies*

$$(2.29) \quad p^x = \nabla_q \psi(q + t\nabla\xi(p^x); x),$$

then we have

$$(2.30) \quad |\mathcal{P}_{t,q}(p; x) - \mathcal{P}_{t,q}(p^x; x)| \leq C |p - p^x|_{L^{\infty}}^2, \quad \forall p \in \mathcal{Q}_{\infty, \leq 1}.$$

Proof. For brevity of notation, we write $\psi(\cdot) = \psi(\cdot; x)$. Let $p \in \mathcal{Q}_{\infty, \leq 1}$ and let B be given as in (1.13). Then, as paths, both p and p^x takes value in B . Let $C_1 = 8t^2 \|\nabla\xi|_B\|_{\text{Lip}(B; \mathbb{R}^{D \times D})}$, using (2.29) and the Fréchet differentiability of ψ as in (2.4), we have

$$\begin{aligned} & |\psi(q + t\nabla\xi(p)) - \psi(q + t\nabla\xi(p^x)) - \langle p^x, t\nabla\xi(p) - t\nabla\xi(p^x) \rangle_{L^2}| \\ & \leq 8 |t\nabla\xi(p) - t\nabla\xi(p^x)|_{L^2}^2 \leq C_1 |p - p^x|_{L^{\infty}}^2. \end{aligned}$$

Recall θ from (1.7). For each $a \in \mathbb{R}^{D \times D}$, we interpret $\nabla^2\xi(a)$ as the linear operator on $\mathbb{R}^{D \times D}$ and thus we can interpret $\nabla\theta(a) = a \cdot \nabla^2\xi(a)$ as the linear operator $\mathbb{R}^{D \times D}$ defined by $a' \mapsto a \cdot \nabla^2\xi(a)a'$. Using the smoothness of θ over B , for some constant C_2 only depending on ξ , we also have

$$\left| t \int_0^1 \theta(p) - t \int_0^1 \theta(p^x) - \langle p^x, t\nabla^2\xi(p^x)(p - p^x) \rangle_{L^2} \right| \leq tC_2 |p - p^x|_{L^{\infty}}^2,$$

where $\nabla^2\xi(p^x)(p - p^x)$ is interpreted pointwise as the path $[0, 1] \ni s \mapsto \nabla^2\xi(p^x(s))(p(s) - p^x(s))$. Similarly, for some constant C_3 only depending on ξ , we have

$$|\langle p^x, t\nabla\xi(p) - t\nabla\xi(p^x) \rangle_{L^2} - \langle p^x, t\nabla^2\xi(p^x)(p - p^x) \rangle_{L^2}| \leq tC_3 |p - p^x|_{L^{\infty}}^2.$$

Combining the above three displays, we get (2.30). $\square$

3. SHORT-TIME SMOOTH SOLUTION OF THE HAMILTON–JACOBI EQUATION

We fix $x \in \mathbb{R}^d$. In this section, we are interested in the partial differential equation

$$(3.1) \quad \begin{cases} \partial_t f - \int \xi(\nabla_q f) = 0 & \text{on } (0, +\infty) \times \mathcal{Q}_2 \\ f(0, \cdot) = \psi(\cdot; x). \end{cases}$$

Equation (3.1) is a so-called Hamilton–Jacobi equation. First, we explain the exact meaning of the term $\int \xi(\nabla_q f)$. Here, $\nabla_q f$ is path-valued. More precisely, $\nabla_q f(t, q)$ is equal to some path $p : [0, 1) \rightarrow \mathbb{S}^D$. The nonlinearity $\int \xi(\nabla_q f(t, q))$ is understood as $\int_0^1 \xi(p(s)) ds$.

We define $X(\cdot, \cdot; x) : \mathbb{R}_+ \times \mathcal{Q}_\infty \rightarrow L^\infty$ by

$$(3.2) \quad X(t, q; x) = q - t \nabla \xi(\nabla_q \psi(q; x)).$$

In view of Proposition 2.2 (2), the definition of $X(\cdot, \cdot; x)$ can be extended to $\mathbb{R}_+ \times \mathcal{Q}_1$. However, we restrict the domain to $\mathbb{R}_+ \times \mathcal{Q}_\infty$ for better regularity.

The family curves $t \mapsto X(t, q; x)$ parametrized by q (x is fixed) can be understood as the characteristic lines of equation (3.1). More precisely, it can be checked that if f is a smooth solution of (3.1), then

$$(3.3) \quad \frac{d}{dt} \nabla_q f(t, X(t, q; x); x) = 0.$$

Thanks to this, one can verify by differentiation that we must have

$$(3.4) \quad f(t, X(t, q; x); x) = \mathcal{P}_{t,q}(p; x),$$

where $p = \nabla_q f(t, X(t, q; x); x) = \nabla_q \psi(q; x)$ and $\mathcal{P}_{t,q}$ is the functional defined in (2.28). If the map $q \mapsto X(t, q; x)$ is a bijection for every $t > 0$, then (3.4) uniquely defines a function f and it is possible to verify that $f(\cdot, \cdot; x)$ is a solution of (3.1).

However, at least in finite dimensional spaces, the characteristic lines can cross. So, the function $q \mapsto X(t, q; x)$ is sometimes non-injective. This means that the recipe given by (3.4) to define a differentiable solution may not work. Even worse, Hamilton–Jacobi equations may sometimes not have any differentiable solutions [30, Section 3.2.5.c Example 6]. To restore the existence and uniqueness of solutions, one needs to appeal to the notion of viscosity solution [30, Section 10]. In this setting, the notion of solution is weakened in order to also allow non-differentiable functions to solve the equation. This theory can be extended to infinite dimensional spaces, and it can be shown that (3.1) also admits a unique viscosity solution [24].

The goal of this section is to show that on the time interval $[0, t_\star)$, the characteristic lines do not cross. This allows us to define via (3.4) a differentiable solution $f(\cdot, \cdot; x)$ of (3.1) on the interval $[0, t_\star)$. In addition, by the uniqueness of the viscosity solution, the function $f(\cdot, \cdot; x)$ we build must coincide with the viscosity solution. In the end, the differentiability of the viscosity solution on the interval $[0, t_\star)$ is going to be one of the key ingredients in the verification of (1.20).

Recall the definition of the operator norm in (2.1). We recall that to improve readability, we sometimes abbreviate ∇_q to ∇ .

Lemma 3.1. *Fix any $x \in \mathbb{R}^d$. The function $X(\cdot, \cdot; x) : \mathbb{R}_+ \times \mathcal{Q}_\infty \rightarrow L^\infty$ is Fréchet differentiable. At each $(t, q) \in \mathbb{R}_+ \times \mathcal{Q}_\infty$, the derivative $\nabla_q X(t, q; x)$ is a bounded linear operator from L^∞ to L^∞ with expression*

$$(3.5) \quad \nabla X(t, q; x) = \mathbf{I} - t \nabla^2 \xi(\nabla \psi(q; x)) \nabla^2 \psi(q; x),$$

where $\mathbf{I}: L^\infty \rightarrow L^\infty$ is the identity operator. Moreover, we have

$$(3.6) \quad \|\mathbf{I} - \nabla X(t, q; x)\|_{\text{Op}(L^\infty; L^\infty)} \leq t/t_\star.$$

As a consequence, if $t < t_\star$, then $\nabla X(t, q; x)$ is bijective on L^∞ .

The interpretation of (3.5) is as follows. For every $\kappa \in L^\infty$, $\nabla X(t, q; x)\kappa \in L^\infty$ is the path

$$(3.7) \quad [0, 1] \ni s \mapsto \kappa(s) - t \nabla^2 \xi(p(s))(A\kappa)(s),$$

where $p = \nabla \psi(q; x)$ is a path in $\mathcal{Q}_{\infty, \leq 1}$ (see (2.3)), $\nabla^2 \xi(p(s))$ is a linear operator on $\mathbb{R}^{D \times D}$ (as ξ is a function from $\mathbb{R}^{D \times D}$ to $\mathbb{R}$), $A = \nabla^2 \psi(q; x)$ is a linear operator on L^∞ (see (2.10)), $A\kappa$ is thus a path in L^∞ , and lastly $\nabla^2 \xi(p(s))(A\kappa)(s)$ is the action of the linear operator $\nabla^2 \xi(p(s))$ on the $\mathbb{R}^{D \times D}$ -valued element $(A\kappa)(s)$.

Proof. Since $x \in \mathbb{R}^d$ is fixed, for brevity we drop the x dependence in the notation and write $\psi(q)$, $X(t, q)$ instead of $\psi(q; x)$, $X(t, q; x)$. Since the dependence of $X(t, q)$ on t is trivial, we only show the Fréchet differentiability in q at fixed t .

Let $q, q' \in \mathcal{Q}_\infty$. By (2.9) with $s = \infty$, we have

$$\nabla \xi(\nabla \psi(q')) = \nabla \xi(\nabla \psi(q) + \nabla^2 \psi(q)(q' - q) + p_{\text{error}}),$$

with $|p_{\text{error}}|_{L^\infty} \leq C_0 |q' - q|_{L^\infty}^2$ for some absolute constant C_0 . Let B be as in (1.13) and $C_1 = \|\nabla^3 \xi|_B\|_\infty = \|\nabla^2 \xi|_B\|_{\text{Lip}}$, we have, for $a, a' \in B$,

$$\nabla \xi(a') = \nabla \xi(a) + \nabla^2 \xi(a)(a' - a) + \text{error}_a(a' - a),$$

where $|\text{error}_a(a' - a)| \leq C_1 |a' - a|^2$. By (2.3), $\nabla \psi(q)$ and $\nabla \psi(q')$ are paths taking values in B . Hence, using the above two displays, we have that,

$$\begin{aligned} \nabla \xi(\nabla \psi(q')) &= \nabla \xi(\nabla \psi(q)) + \nabla^2 \xi(\nabla \psi(q')) \nabla^2 \psi(q)(q' - q) \\ &\quad + \nabla^2 \xi(\nabla \psi(q')) p_{\text{error}} + \text{error}_{\nabla \psi(q)}(\nabla \psi(q') - \nabla \psi(q)). \end{aligned}$$

Again by (2.10), the last two terms are bounded in the L^∞ -norm by $|q' - q|_{L^\infty}^2$. This shows the Fréchet differentiability of the nonlinear part in $X(t, \cdot)$ (see (3.2)), which implies the differentiability of $X(t, \cdot)$ and (3.5).

Next, we verify (3.6). Let $\kappa \in L^\infty$ and recall the interpretation of (3.7). Then, we can compute

$$\begin{aligned} |\nabla^2 \xi(\nabla \psi(q')) \nabla^2 \psi(q) \kappa|_{L^\infty} &\stackrel{(2.3)}{\leq} \sup_{a \in B} \|\nabla^2 \xi(a)\|_{\text{Op}(\mathbb{R}^{D \times D}; \mathbb{R}^{D \times D})} |\nabla^2 \psi(q) \kappa|_{L^\infty} \\ (3.8) \quad &\stackrel{(2.10)}{\leq} \sup_{a \in B} \|\nabla^2 \xi(a)\|_{\text{Op}(\mathbb{R}^{D \times D}; \mathbb{R}^{D \times D})} \|\nabla^2 \psi(q)\|_{\text{Op}(L^\infty; L^\infty)} |\kappa|_{L^\infty} \\ &\leq t_\star^{-1} |\kappa|_{L^\infty}. \end{aligned}$$

The last inequality in the display follows from (1.14) once we have shown the following inequality

$$(3.9) \quad \|\nabla^2 \psi(q)\|_{\text{Op}(L^\infty; L^\infty)} \leq \|\nabla \psi\|_{\text{Lip}(\mathcal{Q}_\infty, \mathcal{Q}_\infty)}.$$

By duality between the L^∞ and L^1 norm, the previous display is equivalent to

$$(3.10) \quad \left| \langle \kappa', \nabla^2 \psi(q_n) \kappa \rangle_{L^2} \right| \leq |\kappa'|_{L^1} \|\nabla \psi\|_{\text{Lip}(\mathcal{Q}_\infty, \mathcal{Q}_\infty)} |\kappa|_{L^\infty}, \quad \forall (\kappa, \kappa') \in L^\infty \times L^1.$$

Let us prove this inequality. Recall id is the path $s \mapsto s \text{id}$ and $\text{id} \in S_+^D$ is the $D \times D$ identity matrix. Let $\kappa : [0, 1] \rightarrow S^D$, and $\kappa \in L^\infty$. For now we assume that κ is Lipschitz. For each $n \in \mathbb{N}$, let $q_n = q + n^{-1} \text{id}$. Then, for each $n \in \mathbb{N}$ and any ε we have $q_n + \varepsilon \kappa \in \mathcal{Q}_\infty$ for sufficiently small ε (depending on n and κ). Using the differentiability of ψ in Proposition 2.2 (4) (with $\mathbf{s} = \infty$), for any $\kappa, \kappa' \in L^\infty$, we have

$$\langle \kappa', \nabla^2 \psi(q_n) \kappa \rangle_{L^2} = \lim_{\varepsilon \rightarrow 0} \left\langle \kappa', \frac{\nabla \psi(q_n + \varepsilon \kappa) - \nabla \psi(q_n)}{\varepsilon} \right\rangle_{L^2}.$$

This implies,

$$(3.11) \quad \begin{aligned} \left| \langle \kappa', \nabla^2 \psi(q_n) \kappa \rangle_{L^2} \right| &\leq \lim_{\varepsilon \rightarrow 0} |\kappa'|_{L^1} \left| \frac{\nabla \psi(q_n + \varepsilon \kappa) - \nabla \psi(q_n)}{\varepsilon} \right|_{L^\infty} \\ &\leq |\kappa'|_{L^1} \|\nabla \psi\|_{\text{Lip}(\mathcal{Q}_\infty, \mathcal{Q}_\infty)} |\kappa|_{L^\infty}. \end{aligned}$$

This yields (3.10) but only under the stronger assumption that κ is Lipschitz. Now let $\kappa \in L^\infty$ without assuming that κ is Lipschitz. We let $(\kappa_k)_k$ denote a sequence of approximations of κ defined by mollification. By usual properties of mollifiers, we have $|\kappa_k|_{L^\infty} \leq |\kappa|_{L^\infty}$ and $\kappa_k \rightarrow \kappa$ in L^p for all $p \in [1, +\infty)$ (see e.g. [30, Theorem 7 in Appendix C.5]). Since each κ_k is Lipschitz on $[0, 1]$, as argued in (3.11), we have that

$$\left| \langle \kappa', \nabla^2 \psi(q_n) \kappa_k \rangle_{L^2} \right| \leq |\kappa'|_{L^1} \|\nabla \psi\|_{\text{Lip}(\mathcal{Q}_\infty, \mathcal{Q}_\infty)} |\kappa_k|_{L^\infty}.$$

In addition, since $\nabla^2 \psi(q_n)$ is a bounded operator L^2 according to (2.10) as given in Proposition 2.2 (4), we have that $\nabla^2 \psi(q_n) \kappa_k$ converges to $\nabla^2 \psi(q_n) \kappa$ in L^2 as $k \rightarrow \infty$. By extracting a subsequence, we may assume that this convergence is also pointwise a.e. on $[0, 1]$. Proposition 2.2 (4) also states that $\nabla^2 \psi(q_n)$ is a bounded operator on L^∞ . Hence, these $\nabla^2 \psi(q_n) \kappa_k$ are bounded a.e. uniformly in k , which allows us to apply the dominated convergence theorem to get

$$\lim_{k \rightarrow +\infty} \langle \kappa', \nabla^2 \psi(q_n) \kappa_k \rangle_{L^2} = \langle \kappa', \nabla^2 \psi(q_n) \kappa \rangle_{L^2}.$$

Together, the previous two displays yield

$$\left| \langle \kappa', \nabla^2 \psi(q_n) \kappa \rangle_{L^2} \right| \leq |\kappa'|_{L^1} \|\nabla \psi\|_{\text{Lip}(\mathcal{Q}_\infty, \mathcal{Q}_\infty)} |\kappa|_{L^\infty}.$$

So we have proven (3.10) at $q_n \in \mathcal{Q}_{\infty, \uparrow}$. Using (2.11) from Proposition 2.2 (with $p = q = \infty$) and the convergence of q_n to q in L^∞ , we can obtain (3.9) at q .

Using (3.8) and (3.5), we obtain (3.6). Lastly, the bijectivity is a classical consequence of (3.6) (see [16, Exercise 6.14]). $\square$

Lemma 3.2. *Fix any $x \in \mathbb{R}^d$ and write $\nabla = \nabla_q$. There is a map $Z(\cdot, \cdot; x) : [0, t_\star) \times \mathcal{Q}_\infty \rightarrow \mathcal{Q}_\infty$ such that*

$$(3.12) \quad q = X(t, Z(t, q; x); x), \quad \forall (t, q) \in [0, t_\star) \times \mathcal{Q}_\infty.$$

This in particular implies $X(t, \mathcal{Q}_\infty; x) \supset \mathcal{Q}_\infty$. Moreover, $Z(\cdot, \cdot; x) : [0, t_\star) \times \mathcal{Q}_\infty \rightarrow \mathcal{Q}_\infty$ is Fréchet differentiable everywhere and its derivative at $(t, q) \in [0, t_\star) \times \mathcal{Q}_\infty$ is given by

$$(3.13) \quad \begin{aligned} \nabla Z(t, q; x) &= (\nabla X(t, Z(t, q; x); x))^{-1}, \\ \partial_t Z(t, q; x) &= -(\nabla X(t, Z(t, q; x); x))^{-1} (\nabla \xi(\nabla \psi(q; x))). \end{aligned}$$

Here, $\nabla X(t, Z(t, q; x); x)$ is the derivative of $X(t, \cdot; x)$ evaluated at $Z(t, q; x)$; $(\nabla X(t, Z(t, q; x); x))^{-1}$ is the inverse of the operator $\nabla X(t, Z(t, q; x); x)$ given as in Lemma 3.1; and $(\nabla X(t, Z(t, q; x); x))^{-1} (\nabla \xi(\nabla \psi(q; x)))$ applies the operator at $\nabla \xi(\nabla \psi(q; x)) \in L^\infty$.

Proof. Again, for brevity, we omit the x dependence and write ψ , $X(\cdot, \cdot)$, and $Z(\cdot, \cdot)$ in place of $\psi(\cdot; x)$, $X(\cdot, \cdot; x)$, and $Z(\cdot, \cdot; x)$. Fix any $t \in [0, t_\star)$. Let $q' \in \mathcal{Q}_\infty$ and consider the mapping from $\Phi : \mathcal{Q}_\infty \rightarrow \mathcal{Q}_\infty$ given by

$$\Phi(q) = q' + t \nabla \xi(\nabla \psi(q)), \quad \forall q \in \mathcal{Q}_\infty.$$

By (2.3), this is well-defined. In view of the definition of $t_\star$ in (1.14), we have $\|\Phi\|_{\text{Lip}(\mathcal{Q}_\infty; \mathcal{Q}_\infty)} \leq t/t_\star < 1$. Hence, by the fixed point iteration, there is a unique $q \in \mathcal{Q}_\infty$ such that $q = \Phi(q)$ and thus $q' = X(t, q)$. Hence, for each $q' \in \mathcal{Q}_2$, we can define $Z(t, q') = q$. This immediately gives (3.12).

Next, we want to verify that Z is Fréchet differentiable. For this purpose, we first verify that Z is Lipschitz continuous on $[t, t_1] \times \mathcal{Q}_\infty$ for all $t_1 < t_\star$. From

$$(3.14) \quad q \stackrel{(3.12)}{=} X(t, Z(t, q)) \stackrel{(3.2)}{=} Z(t, q) - t \nabla \xi(\nabla \psi(Z(t, q))),$$

and the definition of $t_\star$ in (1.14), we can deduce

$$\sup_{t \in [0, t_1]} \|Z(t, \cdot)\|_{\text{Lip}(\mathcal{Q}_\infty; \mathcal{Q}_\infty)} < \infty.$$

Similarly, from (3.14) and the boundedness of $\nabla \psi$ as in (2.3), we can deduce the Lipschitzness of $Z(\cdot, q)$ on $[0, t_1]$ uniformly in each $q \in \mathcal{Q}_\infty$. Together, we have the joint Lipschitz of Z on $[t, t_1] \times \mathcal{Q}_\infty$.

Now, we are ready to prove the differentiability of Z . Fix any $(t, q) \in [0, t_\star) \times \mathcal{Q}_\infty$ and write $p = Z(t, q)$. As (t', q') approaching (t, q) , we can use the Fréchet differentiability of X from Lemma 3.1 and (3.12) (at both (t, q) and (t', q')) to see

$$\begin{aligned} |q' - q - \partial_t X(t, p)(t' - t) - \nabla X(t, p)(Z(t', q') - Z(t, q))|_{L^\infty} \\ = o(|t' - t| + |Z(t', q') - Z(t, q)|_{L^\infty}). \end{aligned}$$

Notice that $\partial_t X(t, p) = -\nabla \xi(\nabla \psi(q))$ simply due to (3.2). By the Lipschitzness of Z , the error is controlled by $o(|t' - t| + |q' - q|_{L^\infty})$. Using this and the

invertibility of $\nabla X(t, p)$ given by Lemma 3.1, we can see that Z is Fréchet differentiable at (t, q) and has its derivative given as in (3.13). $\square$

Associated with the characteristic line $X(\cdot, \cdot; x)$ is the function $U(\cdot, \cdot; x) : \mathbb{R}_+ \times \mathcal{Q}_\infty \rightarrow \mathbb{R}$ defined by, for every $(t, q) \in \mathbb{R}_+ \times \mathcal{Q}_\infty$,

(3.15)

$$U(t, q; x) = \psi(q; x) - t \langle \nabla \xi(\nabla \psi(q; x)), \nabla \psi(q; x) \rangle_{L^2} + t \int \xi(\nabla \psi(q; x)).$$

Also, recall $\mathcal{P}_{t,q}(\cdot; x)$ defined in (2.28).

Proposition 3.3. *Fix any $x \in \mathbb{R}^d$. Let $f(\cdot, \cdot; x)$ be defined by*

$$(3.16) \quad f(t, q; x) = U(t, Z(t, q; x); x), \quad \forall (t, q) \in [0, t_\star) \times \mathcal{Q}_\infty.$$

Then, $f(\cdot, \cdot; x)$ is twice Fréchet differentiable everywhere on $[0, t_\star) \times \mathcal{Q}_\infty$ and for every $(t, q) \in [0, t_\star) \times \mathcal{Q}_\infty$ we have

$$\partial_t f(t, q) - \int \xi(\nabla_q f(t, q)) = 0.$$

Furthermore, writing $p = \nabla_q f(t, q; x)$, we have

$$(3.17) \quad p = \nabla_q \psi(q + t \nabla \xi(p); x),$$

$$(3.18) \quad f(t, q; x) = \mathcal{P}_{t,q}(p; x).$$

Proof. Again, for brevity we write $f(\cdot, \cdot) = f(\cdot, \cdot; x)$, $\psi = \psi(\cdot; x)$, and $U(\cdot, \cdot) = U(\cdot, \cdot; x)$. We first verify the initial condition. Since $U(0, \cdot) = \psi$ and $Z(0, \cdot)$ is the identity map (due to (3.2) and (3.12)), we get $f(0, \cdot) = \psi$ as desired.

From the definition of f in (3.16), that of U in (3.15), Fréchet differentiability of Z in Lemma 3.2, and regularity of ψ from Proposition 2.2, it is straightforward to check that f is Fréchet differentiable everywhere on $[0, t_\star) \times \mathcal{Q}_\infty$. In the following, we first verify that f satisfies the equation (3.1), in the process of which we will also show that f is in fact twice differentiable.

For (t, q) clear from the context and any relevant function h , we write ∇h^Z or $\partial_t h^Z$ for the evaluation of ∇h or $\partial_t h$ at the spatial variable $Z(t, q)$. For brevity, we also write

$$g = \nabla \xi(\nabla \psi(\cdot)), \quad \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2}.$$

In the following computations, we keep $(t, q) \in [0, t_\star) \times \mathcal{Q}_\infty$ implicit. We start with some basics:

$$(3.19) \quad \begin{aligned} \nabla U &\stackrel{(3.15)}{=} \nabla \psi - t (\nabla g)^\top (\nabla \psi), \\ \nabla X &\stackrel{(3.2)}{=} \mathbf{I} - t \nabla g, \\ \nabla f &\stackrel{(3.16)}{=} (\nabla Z)^\top \nabla U^Z, \end{aligned}$$

where $\mathbf{I}$ is identity operator on L^∞ . The first two relations give

$$(3.20) \quad \nabla U = (\nabla X)^\top \nabla \psi.$$

By (3.12), we also have

$$(\nabla Z(t, q))(\nabla X(t, Z(t, q))) = \mathbf{I},$$

where the product on the left is the composition of two linear operators on L^∞ . The above two displays along with (3.19) imply

$$(3.21) \quad \nabla f = \nabla \psi^Z.$$

Next, we compute the derivative of U in t . We start with

$$(3.22) \quad \partial_t X \stackrel{(3.2)}{=} -\nabla \xi(\nabla \psi),$$

$$\partial_t f \stackrel{(3.16)}{=} \partial_t U^Z + \langle \nabla U^Z, \partial_t Z \rangle$$

$$(3.23) \quad \stackrel{(3.15)(3.20)}{=} -\langle \nabla \xi(\nabla \psi^Z), \nabla \psi^Z \rangle + \int \xi(\nabla \psi^Z) + \langle (\nabla X^Z)^\top \nabla \psi^Z, \partial_t Z \rangle.$$

From (3.12), we get

$$\partial_t X^Z + (\nabla X^Z) \partial_t Z = 0.$$

Using this and (3.22) to see that the first and third terms in (3.23) cancel each other, we get

$$\partial_t f = \int \xi(\nabla \psi^Z).$$

This along with (3.21) implies that f satisfies the equation (3.1) everywhere on $[0, t_\star) \times \mathcal{Q}_\infty$ in the classical sense. Moreover, we can also infer from the above display, (3.21), the differentiability of Z in Lemma 3.2, and the regularity of ψ in Proposition 2.2 that f is twice Fréchet differentiable.

Lastly, we verify (3.17) and (3.18). We start with rewriting (3.21) as

$$\nabla f(t, q) = \nabla \psi(Z(t, q)).$$

Using (3.2) and (3.12), we can get

$$Z(t, q) = q + t \nabla \xi(\nabla \psi(Z(t, q))).$$

Combining these two displays, we get (3.17). Also, inserting these two relations into (3.16) and recalling the definition of U in (3.15) and $\mathcal{P}_{t,q}$ in (2.28), we can verify (3.18). $\square$

Lemma 3.4. *For every $x \in \mathbb{R}^d$ and $(t, q) \in [0, t_\star) \times \mathcal{Q}_\infty$, there is a unique solution p to $p' = \nabla_q \psi(q + t \nabla \xi(p'); x)$ over $p' \in \mathcal{Q}_\infty$.*

Proof. By (2.3) in Proposition 2.2, $p \mapsto \nabla_q \psi(q + t \nabla \xi(p); x)$ is a well-defined function from $\mathcal{Q}_\infty$ to $\mathcal{Q}_\infty$. By the definition of $t_\star$ in (1.14), we have

$$(3.24) \quad |\nabla_q \psi(q + t \nabla \xi(p); x) - \nabla_q \psi(q + t \nabla \xi(p'); x)|_{L^\infty} \leq t t_\star^{-1} |p - p'|_{L^\infty}, \quad \forall p, p' \in \mathcal{Q}_\infty.$$

Then, the desired result follows from the Banach fixed point theorem. $\square$

Finally, for an approximation argument later, we need the following lemma, which establishes the Lipschitz continuity of f under the L^1 topology.

Lemma 3.5. *Fix any $x \in \mathbb{R}^d$. Then, $f(\cdot, \cdot; x) : [0, t_*) \times \mathcal{Q}_\infty \rightarrow \mathbb{R}$ is Lipschitz continuous with respect to the L^1 -norm. More precisely, we have for every $(t, q), (t', q') \in [0, t_*) \times \mathcal{Q}_\infty$,*

$$(3.25) \quad |f(t, q, x) - f(t', q', x)| \leq |q - q'|_{L^1} + |t - t'| \sup_{|a| \leq 1} |\xi(a)|.$$

In particular, this means that f can be extended by continuity to a function on $[0, t_) \times \mathcal{Q}_1$.*

Proof. We write $f(\cdot, \cdot) = f(\cdot, \cdot; x)$, $\psi(\cdot) = \psi(\cdot; x)$, and $L^r = L^r([0, 1]; \mathbb{S}^D)$ for brevity. According to Proposition 3.3, f is Fréchet differentiable on $[0, t_*) \times \mathcal{Q}_\infty$ and we have

$$\nabla_q f(t, q) = \nabla_q \psi(q + t \nabla f(t, q)) \quad \text{and} \quad \partial_t f(t, q) = \int \xi(\nabla_q f(t, q)).$$

The first line together with (2.3) ensures that $|\nabla_q f(t, q)|_{L^\infty} \leq 1$. Plugging this in the second line yields $|\partial_t f(t, q)| \leq \sup_{|a| \leq 1} |\xi(a)|$. Thanks to this, we have

$$\begin{aligned} |f(t, q) - f(t', q')| &= \left| \int_0^1 \frac{d}{ds} f(s(t, q) + (1-s)(t', q')) ds \right| \\ &= \left| \int_0^1 \langle \nabla f(s(t, q) + (1-s)(t', q')), q - q' \rangle_{L^2} \right. \\ &\quad \left. + \partial_t f(s(t, q) + (1-s)(t', q'))(t - t') ds \right| \\ &\leq |q - q'|_{L^1} \sup_s |\nabla f(s(t, q) + (1-s)(t', q'))|_{L^\infty} + |t - t'| \sup_{|a| \leq 1} |\xi(a)| \\ &\leq |q - q'|_{L^1} + |t - t'| \sup_{|a| \leq 1} |\xi(a)|. \end{aligned}$$

□

4. PROOF OF THE MAIN RESULTS

4.1. Preliminaries. From [22, Proposition 3.1], we recall that $\overline{F}_N^{m \rightarrow x \cdot m}$ defined in (1.11) is Lipschitz in (t, q) uniformly in (N, x) . More precisely, for every $t, t' \in \mathbb{R}_+$ and $q, q' \in \mathcal{Q}_1$, we have

$$(4.1) \quad |\overline{F}_N^{m \rightarrow x \cdot m}(t, q) - \overline{F}_N^{m \rightarrow x \cdot m}(t', q')| \leq |q - q'|_{L^1} + |t - t'| \sup_{|a| \leq 1} |\xi(a)|.$$

According to the Arzelà–Ascoli theorem, the sequence $(\overline{F}_N^{m \rightarrow x \cdot m})_{N \in \mathbb{N}}$ admits subsequential limits in the topology of local uniform convergence on $\mathbb{R}_+ \times \mathcal{Q}_r$ for any $r \in [1, \infty]$.

Recall the definition of $\mathcal{P}_{t,q}$ from (2.28). The following proposition is partly inspired by [28, Proposition 1.3].

Proposition 4.1. *Let $x \in \mathbb{R}^d$. Then, for every $(t, q) \in [0, t_*) \times \mathcal{Q}_2$, we have*

$$\lim_{N \rightarrow \infty} \overline{F}_N^{m \rightarrow x \cdot m}(t, q) = \mathcal{P}_{t,q}(p; x),$$

where p is the unique solution to $p = \nabla_q \psi(q + t \nabla \xi(p); x)$ given by Lemma 3.4.

To prove this proposition, we verify that any subsequential limit of the sequence $(\bar{F}_N^{m \mapsto x \cdot m}(\cdot, q))_N$ is differentiable on $(0, t_*)$ and that its derivative can be uniquely characterized. This will allow us to show that the sequence $(\bar{F}_N^{m \mapsto x \cdot m}(\cdot, q))_N$ converges and to identify its limit. To proceed, we will need the following lemma, which states that the free energy $\bar{F}_N^{m \mapsto x \cdot m}(\cdot, q)$ is locally semi-concave in $t \in (0, +\infty)$, uniformly in N , q , and x .

Lemma 4.2. *For every $c > 0$ there is a constant $C > 0$ such that for every $N \in \mathbb{N}$, every $x \in \mathbb{R}^d$ and every $q \in \mathcal{Q}_1$ we have for all $t, t' \in [c, +\infty)$ and $\lambda \in [0, 1]$,*

$$(1 - \lambda)\bar{F}_N(t) + \lambda\bar{F}_N(t') - \bar{F}_N((1 - \lambda)t + \lambda t') \leq C\lambda(1 - \lambda)(t - t')^2$$

where we used the shorthand $\bar{F}_N(\cdot) = \bar{F}_N^{m \mapsto x \cdot m}(\cdot, q)$.

Proof. Recall the Hamiltonian $H_{N;t,q}^G(\sigma, \alpha)$ from (1.10). For each $s \geq 0$, we set $H_s(\sigma, \alpha) = H_{N;s^2,q}^{m \mapsto x \cdot m}(\sigma, \alpha)$ and define

$$G(s) = -\frac{1}{N} \mathbb{E} \log \iint \exp(H_s(\sigma, \alpha)) dP_1^{\otimes N}(\sigma) d\mathfrak{R}(\alpha).$$

In view of (1.11), we have $\bar{F}_N(t) = G(\sqrt{t})$. This implies,

$$\bar{F}_N''(t) = \frac{G''(\sqrt{t})}{2\sqrt{t}} - \frac{G'(\sqrt{t})}{t^{\frac{3}{2}}}.$$

The local semi-concavity of $\bar{F}_N$ can be reformulated as $\bar{F}_N''$ being upper-bounded on $[c, +\infty)$. The function G' is bounded, this can be easily shown using Gaussian integration by parts. Hence, it suffices to show that the G'' is upper-bounded.

Let $R = \lambda(1 - \lambda)(s - s')^2 N \xi(\sigma \sigma^\top / N)$. Using the definition of G , we have

$$\begin{aligned} (1 - \lambda)G(s) + \lambda G(s') &= -\frac{1}{N} \mathbb{E} \log (\dots)^{1-\lambda} (\dots)^\lambda \\ &\stackrel{\text{Jensen}}{\leq} -\frac{1}{N} \mathbb{E} \log \iint e^{(1-\lambda)H_s(\sigma, \alpha) + \lambda H_{s'}(\sigma, \alpha)} dP_1^{\otimes N}(\sigma) d\mathfrak{R}(\alpha) \\ &= -\frac{1}{N} \mathbb{E} \log \iint e^{H_{(1-\lambda)s + \lambda s'}(\sigma, \alpha) - R} dP_1^{\otimes N}(\sigma) d\mathfrak{R}(\alpha). \end{aligned}$$

Setting $M = \max_{a \in \mathbb{S}_+^D: |a| \leq 1} |\xi(a)|$, we have $R \leq \lambda(1 - \lambda)(s - s')^2 NM$ and thus

$$(1 - \lambda)G(s) + \lambda G(s') \leq G((1 - \lambda)s + \lambda s') + \lambda(1 - \lambda)(s - s')^2 M.$$

Setting $\lambda = \frac{1}{2}$ and $s' = s + \varepsilon$ and sending $\varepsilon \rightarrow 0$ in this display, we get $G''(s) \leq 2M$, completing the proof as explained before. $\square$

Proof of Proposition 4.1. Since x is fixed, we omit it from the notation and we make the dependence of the fixed point p in (t, q) explicit by writing it $p^{t,q}$.

The Arzelà–Ascoli theorem gives a subsequence along which the free energy converges to some limit $f : \mathbb{R}_+ \times \mathcal{Q}_2 \rightarrow \mathbb{R}$. We recall from [22, Proposition 5.3]

that f is Gateaux differentiable jointly in t and q on a dense subset of $\mathbb{R}_+ \times \mathcal{Q}_2$ (in the L^2 -topology). We denote this set of Gateaux differentiable point as G . For each $(t, q) \in G$, [22, Proposition 7.2] gives $\partial_t f(t, q) = \int_0^1 \xi(p^{t,q})$ for some $p^{t,q} \in \mathcal{Q}_{\infty, \leq 1}$ that satisfies the fixed point equation $p' = \nabla_q \psi(q + t \nabla(p'))$. Lemma 3.4 implies $p^{t,q} = p^{t,q}$ whenever $t < t_*$ and thus

$$\partial_t f(t, q) = \int_0^1 \xi(p^{t,q}), \quad \forall (t, q) \in G, \quad t < t_*.$$

Now, fix any $(t, q) \in (0, t_*) \times \mathcal{Q}_2$. For s sufficiently small, there is $(t_s, q_s) \in G$ satisfying $|t - t_s| + |q - q_s|_{L^2} \leq s^2$. From this and the Lipschitz continuity of f inherited from (4.1), we have that there exists some absolute constants $C_1, C_2 > 0$ such that for sufficiently small s ,

$$\begin{aligned} s^{-1} (f(t + s, q) - f(t, q)) &\leq s^{-1} (f(t_s + s, q_s) - f(t_s, q_s)) + C_1 s \\ &\leq \int_0^1 \xi(p^{t_s, q_s}) + C_2 s, \end{aligned}$$

where we used the local semi-concavity of f inherited from that enjoyed by the free energy as given in Lemma 4.2 below. For the converse bound, we choose (t'_s, q'_s) satisfying $(t'_s + s, q'_s) \in G$ and $|t - t'_s| + |q - q'_s|_{L^2} \leq s^2$. By similar arguments, we get that there exists some absolute constants $C_3, C_4 > 0$, such that for sufficiently small s ,

$$\begin{aligned} s^{-1} (f(t + s, q) - f(t, q)) &\geq s^{-1} (f(t'_s + s, q'_s) - f(t'_s, q'_s)) - C_3 s \\ &\geq \int_0^1 \xi(p^{t'_s + s, q'_s}) - C_4 s. \end{aligned}$$

By a similar argument as in (4.3), we can see that the map $[0, t_*) \times \mathcal{Q}_2 \ni (t, q) \mapsto p^{t,q}$ is continuous in L^∞ . Therefore, sending $s \rightarrow 0$ in the above two displays, we obtain the differentiability of $f(\cdot, q)$ and

$$\partial_t f(t, q) = \int_0^1 \xi(p^{t,q}), \quad \forall (t, q) \in (0, t_*) \times \mathcal{Q}_2.$$

Hence, for every $(t, q) \in (0, t_*) \times \mathcal{Q}_2$, we conclude

$$f(t, q) = \psi(q) + \int_0^t \left(\int_0^1 \xi(p^{t,q}) \right) dt.$$

This uniquely characterizes f , and proves that it is independent of the chosen subsequence. Therefore, we conclude that $\bar{F}_N$ converges at every $(t, q) \in (0, t_*) \times \mathcal{Q}_2$. By [22, Proposition 7.3], the limit at (t, q) is equal to $\mathcal{P}_{t,q}(p; x)$ for some $p \in \mathcal{Q}_{\infty, \leq 1}$ satisfying the fixed point equation $p = \nabla_q \psi(q + t \nabla(p)); x$. Again, Lemma 3.4 ensures that $p = p_{q,t}$, completing the proof. $\square$

Corollary 4.3. *Fix any $x \in \mathbb{R}^d$. Then, for every $(t, q) \in [0, t_*) \times \mathcal{Q}_\infty$, we have*

$$(4.2) \quad \lim_{N \rightarrow \infty} \bar{F}_N^{m \mapsto x \cdot m}(t, q) = f(t, q; x),$$

where $f(\cdot, \cdot; x)$ is the function constructed in Proposition 3.3.

Proof. This follows from Proposition 4.1 and Proposition 3.3. $\square$

Lemma 4.4. *Fix any $(t, q) \in [0, t_*) \times \mathcal{Q}_\infty$ the function $x \mapsto f(t, q; x)$ is Lipschitz, concave, and continuously differentiable.*

Proof. Since we have $f(t, q; x) = \lim_{N \rightarrow +\infty} \overline{F}_N^{m \mapsto x \cdot m}(t, q)$ for every $x \in \mathbb{R}^d$ due to Corollary 4.3, the concavity and Lipschitzness of $f(t, q; \cdot)$ follows from that of $x \mapsto \overline{F}_N^{m \mapsto x \cdot m}(t, q)$, which are evident from its definition in (1.11) and the boundedness of m_N (see (1.8)). It remains to prove the differentiability. Once this is done, the continuity of the derivative follows directly since $f(t, q; \cdot)$ is concave. Recall $\mathcal{P}_{t,q}$ defined in (2.28). Since t and q are fixed, we write $\mathcal{P}(\cdot; x) = \mathcal{P}_{t,q}(\cdot; x)$ for brevity. By Proposition 3.3, $\nabla_q f(t, q; x)$ exists for every $x \in \mathbb{R}^d$ and we denote it by $p^x = \nabla_q f(t, q; x)$. By (2.3), we have $p^x \in \mathcal{Q}_{\infty, \leq 1}$ defined in (2.27). By (3.17) and (3.18) from the same proposition, we have

$$f(t, q; x) = \mathcal{P}(p^x; x) \quad \text{and} \quad p^x = \nabla_q \psi(q + t \nabla \xi(p^x); x), \quad \forall x \in \mathbb{R}^d.$$

Using the latter relation, for $x, x' \in \mathbb{R}^d$ we can compute

$$\begin{aligned} |p^x - p^{x'}|_{L^\infty} &\leq \left| \nabla_q \psi(q + t \nabla \xi(p^x); x) - \nabla_q \psi(q + t \nabla \xi(p^{x'}); x) \right|_{L^\infty} \\ &\quad + \left| \nabla_q \psi(q + t \nabla \xi(p^{x'}); x) - \nabla_q \psi(q + t \nabla \xi(p^{x'}); x') \right|_{L^\infty} \\ (4.3) \quad &\stackrel{(1.14)(2.5)}{\leq} tt_*^{-1} |p^x - p^{x'}|_{L^\infty} + 4|h|_{L^\infty} |x - x'|. \end{aligned}$$

Due to $t < t_*$, we can see that $x \mapsto p^x$ is Lipschitz from $\mathbb{R}^d$ to $\mathcal{Q}_{\infty, \leq 1}$. Using (2.7) from Proposition 2.2 (3), we have

$$\left| \mathcal{P}(p^{x'}; x') - \mathcal{P}(p^{x'}; x) - (x' - x) \cdot \nabla_x \psi(q + t \nabla \xi(p^{x'}); x) \right| \leq |h|_{L^\infty}^2 |x - x'|^2.$$

The Lipschitzness of $x \mapsto p^x$ further implies that as x' tends to x , we have

$$\mathcal{P}(p^{x'}; x') - \mathcal{P}(p^{x'}; x) = (x' - x) \cdot \nabla_x \psi(q + t \nabla \xi(p^x); x) + o(|x - x'|).$$

Furthermore, using (2.30) from Lemma 2.3, we have

$$|\mathcal{P}(p^{x'}; x) - \mathcal{P}(p^x; x)| \leq C |p^{x'} - p^x|_{L^\infty}^2.$$

Combining the previous two displays; we obtain

$$\begin{aligned} \mathcal{P}(p^{x'}; x') - \mathcal{P}(p^x; x) &= \mathcal{P}(p^{x'}; x') - \mathcal{P}(p^{x'}; x) + \mathcal{P}(p^{x'}; x) - \mathcal{P}(p^x; x) \\ &= (x' - x) \cdot \nabla_x \psi(q + t \nabla \xi(p^x); x) + o(|x - x'|) + o(|p^x - p^{x'}|_{L^\infty}). \end{aligned}$$

Observe that the combined error is $o(|x - x'|)$ due to the Lipschitzness of $x \mapsto p^x$ proved above. Recall $\mathcal{P}(p^x; x) = f(t, q; x)$ and we conclude from the above display that $f(t, q; \cdot)$ is differentiable at every x . Since $f(t, q; \cdot)$ is also concave, then it is a classical result from convex analysis that differentiability implies continuous differentiability. $\square$

4.2. Limit free energy and the large deviation principle. Recall the definition of the Gibbs measure $\langle \cdot \rangle_{N;t,q}^G$ from (1.22) and recall that given a random variable $X : \Omega \rightarrow \mathfrak{X}$ and $\omega \in \Omega$ we let $X^\omega \in \mathfrak{X}$ denote the realization of the random variable X on the event ω . Also, recall the definition of $f^*(t, q; m)$ from (1.23).

Lemma 4.5. *Let $(t, q) \in [0, t_*) \times \mathcal{Q}_\infty$. There exists a full-measure subset $\Omega' \subseteq \Omega$ such that, for every $\omega \in \Omega'$, m_N under $\langle \cdot \rangle_{N;t,q}^{0,\omega}$ satisfies a large deviation principle with rate function $I_{t,q}$ defined by*

$$(4.4) \quad I_{t,q}(m) = f^*(t, q; m) - f(t, q; 0), \quad \forall m \in \mathbb{R}^d.$$

Proof. Using Corollary 4.3 and the concentration of free energy as in Remark 1.5, we can find a full-measure subset $\Omega' \subseteq \Omega$ such that, for every $\omega \in \Omega'$ and every $y \in \mathbb{Q}^d$,

$$\lim_{N \rightarrow \infty} F_N^{m \mapsto y \cdot m, \omega}(t, q) = f(t, q; y).$$

Henceforth, we fix $\omega \in \Omega'$ and define for each $y \in \mathbb{R}^d$,

$$\Lambda_{N;t,q}(y) = \frac{1}{N} \log \left\langle e^{Ny \cdot m_N} \right\rangle_{N;t,q}^{0,\omega}.$$

Notice that $\Lambda_{N;t,q}(y) = -F_N^{m \mapsto y \cdot m, \omega}(t, q) + F_N^{0,\omega}(t, q)$. Hence, we have for every $y \in \mathbb{Q}^d$

$$(4.5) \quad \lim_{N \rightarrow \infty} \Lambda_{N;t,q}(y) = -f(t, q; y) + f(t, q; 0).$$

Since $\Lambda_{N;t,q}$ is Lipschitz uniformly in N and $f(t, q; \cdot)$ is also Lipschitz, the previous display in fact holds for every $y \in \mathbb{R}^d$. Finally, since by Lemma 4.4, $f(t, q; \cdot)$ is differentiable everywhere, the Gärtner–Ellis theorem (e.g., see [26, Theorem 2.3.6]) guarantees that m_N satisfies a large deviation principle under $\langle \cdot \rangle_{N;t,q}^{0,\omega}$ with rate function

$$I_{t,q}(m) = \sup_{y \in \mathbb{R}^d} \left\{ y \cdot m - \lim_{N \rightarrow +\infty} \Lambda_{N;t,q}(y) \right\} \stackrel{(4.5)(1.23)}{=} f^*(t, q; m) - f(t, q; 0).$$

□

Lemma 4.6. *Let $(t, q) \in [0, t_*) \times \mathcal{Q}_\infty$. For every continuous function $G : \mathbb{R}^d \rightarrow \mathbb{R}$ and every $\omega \in \Omega'$ (as given in Lemma 4.5), we have*

$$(4.6) \quad \lim_{N \rightarrow +\infty} \frac{1}{N} \log \left\langle e^{NG(m_N)} \right\rangle_{N;t,q}^{0,\omega} = \sup_{m \in \mathbb{R}^d} \{G(m) - I_{t,q}(m)\}.$$

Proof. Since m_N is bounded uniformly in N , so is $G(m_N)$, which allows us to apply Varadhan's lemma (e.g., see [26, Theorem 4.3.1]) to deduce the desired result from the large deviation principle in Lemma 4.5. □

Proof of Theorem 1.4. Let Ω' be given as in Lemma 4.5. Notice that, for every $\omega \in \Omega'$,

$$\frac{1}{N} \log \left\langle e^{NG(m_N)} \right\rangle_{N;t,q}^{0,\omega} = -F_N^{G,\omega}(t, q) + F_N^{0,\omega}(t, q).$$

Using Lemma 4.6, (4.4), and $\lim_{N \rightarrow \infty} F_N^{0,\omega}(t, q) = f(t, q; y)$ due to Corollary 4.3, we get that for every $\omega \in \Omega'$

$$\lim_{N \rightarrow \infty} F_N^{G,\omega} = \inf_{m \in \mathbb{R}^d} \{-G(m) + f^*(t, q; m)\}.$$

Finally, using the definition of $f^*(t, q; m)$ in (1.23), we observe that the right-hand side in the previous display is equal to the right-hand side in (1.17). Then, the desired result follows from this and the concentration of free energy as in Remark 1.5. $\square$

Proof of Theorem 1.8. Let Ω' be given as in Lemma 4.5. Fix any continuous function $G : \mathbb{R}^d \rightarrow \mathbb{R}$ and recall $I_{t,q}^G$ from (1.24). Let $\Gamma : \mathbb{R}^d \rightarrow \mathbb{R}$ be any bounded continuous function. Since we have,

$$\frac{1}{N} \log \left\langle e^{N\Gamma(m_N)} \right\rangle_{N;t,q}^{G,\omega} = \frac{1}{N} \log \left\langle e^{N(G+\Gamma)(m_N)} \right\rangle_{N;t,q}^{0,\omega} - \frac{1}{N} \log \left\langle e^{NG(m_N)} \right\rangle_{N;t,q}^{0,\omega}.$$

It follows from Lemma 4.6 along with (4.4) that for every $\omega \in \Omega'$,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \log \left\langle e^{N\Gamma(m_N)} \right\rangle_{N;t,q}^{G,\omega} &= \sup \{G + \Gamma - I_{t,q}\} - \sup \{G - I_{t,q}\} \\ &\stackrel{(1.24)(4.4)}{=} \sup \{\Gamma - I_{t,q}^G\}. \end{aligned}$$

Since this holds for any bounded continuous Γ , we can apply a variant of Bryc's inverse Varadhan lemma (e.g., see the " $\Leftarrow$ " direction in [26, Theorem 4.4.13]) which guarantees that m_N satisfies a large deviation principle under $\langle \cdot \rangle_{N;t,q}^{G,\omega}$ with rate function $I_{t,q}^G$. $\square$

4.3. Replica symmetric form. Recall from (1.25) the notation of a constant path $\hat{y}$ given any $y \in S^D$. In this section, we also write $(y)^\wedge = \hat{y}$. For $\kappa \in L^1$, we define $\tilde{\kappa} = (\kappa)^\vee = \int_0^1 \kappa(r) dr \in S^D$. Then, for every $y \in S^D$ and $\kappa \in L^1$, we have the following simple identities:

$$(4.7) \quad y \cdot \tilde{\kappa} = \langle \hat{y}, \kappa \rangle_{L^2} \quad \text{and} \quad (\hat{y})^\vee = y.$$

Recall $\psi(\cdot, \cdot)$ in (1.12) and $\phi(\cdot, \cdot)$ in (1.27). Also, recall the relation $\phi(y; x) = \psi(\hat{y}; x)$ from (1.34). To prove Theorem 1.9, we show in Proposition 4.8 that for every $x \in \mathbb{R}^D$, the function $(t, y) \mapsto f(t, \hat{y}; x)$ satisfies a *finite* dimensional Hamilton–Jacobi on $[0, t_\star) \times S_+^D$. To do so, we need to recall a modified version of [23, Lemma 5.12]. This lemma relies on the assumption that the support of P_1 spans $\mathbb{R}^D$.

Lemma 4.7 ([23]). *Fix any $x \in \mathbb{R}^d$. Let $q \in \mathcal{Q}_\infty$ and set $p = \nabla_q \psi(q; x)$. For $0 < s < s' < 1$, we have $p(s') \neq p(s)$ if and only if $q(s') \neq q(s)$.*

In [23], the paths (such as p, q) are taken to be their left-continuous versions. But the result remains the same here for right-continuous ones (see the definition of $\mathcal{Q}$ in Section 1.2). The setting in [18] concerns possibly non-convex vector spin glass models without extra random spins and Curie–Weiss-type interactions. Namely, the function G is chosen to be the null

function in (1.10) and (1.11). We explain that the result in [18] can be adapted straightforwardly.

Sketch of Proof of Lemma 4.7. The proof of Lemma 4.7 and most of the other results in [18] are based on the analysis of the Parisi PDE and the associated SDE. The Parisi PDE gives a concrete way to compute the integration w.r.t. the Gaussian field $(w^q(\alpha))_\alpha$ indexed by the cascade variable α in $\psi(q; 0)$ (see (1.12)). Now, with the additional field $x \cdot h(\tau, \chi_1)$, the initial condition (see [23, (4.2)]) of the Parisi PDE becomes the function

$$\mathbb{R}^D \ni z \mapsto \mathbb{E}_{\chi_1} \log \int \exp(\tau \cdot z - \tau \tau^\top \cdot a + x \cdot h(\tau, \chi_1)) dP_1(\tau),$$

where $a \in \mathbb{S}_+^D$ is some constant. The version in [23, (4.2)] corresponds to the case $x = 0$. The analysis in [23] only uses the smoothness, Lipschitzness, and convexity of the initial condition, which are still preserved despite the presence of $h(\tau, \chi_1)$. Therefore, working with the above initial condition, we can straightforwardly adapt [23, Lemma 5.12] to Lemma 4.7 here. $\square$

Proposition 4.8. *Let $x \in \mathbb{R}^d$ and let $f(\cdot, \cdot; x)$ be given as in Proposition 3.3. Define $g(\cdot, \cdot; x) : [0, t_\star) \times \mathbb{S}_+^D \rightarrow \mathbb{R}$ by*

$$(4.8) \quad g(t, y; x) = f(t, \hat{y}; x).$$

The function $g(\cdot, \cdot; x)$ is differentiable on $[0, t_\star) \times \mathbb{S}_+^D$ and satisfies

$$(4.9) \quad \begin{cases} \partial_t g(t, y; x) - \xi(\nabla_y g(t, y; x)) = 0 & \text{for } (t, y) \in (0, t_\star) \times \mathbb{S}_+^D, \\ g(0, y; x) = \phi(y; x) & \text{for } y \in \mathbb{S}_+^D. \end{cases}$$

Moreover, at every $(t, y) \in [0, t_\star) \times \mathbb{S}_+^D$ we have,

$$(4.10) \quad \partial_t g(t, y; x) = \partial_t f(t, \hat{y}; x) \quad \text{and} \quad (\nabla_y g(t, y; x))^\wedge = \nabla_q f(t, \hat{y}; x).$$

Finally, setting $z = \nabla_y g(t, y; x)$, we have that z is the unique solution to $z' = \nabla_y \phi(y + t \nabla \xi(z'); x)$ over $z' \in \mathbb{S}_+^D$ and we have

$$(4.11) \quad g(t, y; x) = \phi(y + t \nabla \xi(z); x) - t \theta(z).$$

Proof. For brevity, we drop x from the notation and write $g = g(\cdot, \cdot; x)$, $f = f(\cdot, \cdot; x)$, and $\psi = \psi(\cdot; x)$. Fix any $(t, y) \in [0, t_\star) \times \mathbb{S}_+^D$. From the definition of g , it is easy to see the first relation in (4.10). To see the second one, let $y' \in \mathbb{S}_+^D$ and we have

$$\begin{aligned} \frac{d}{d\varepsilon} g(t, y + \varepsilon(y' - y)) \Big|_{\varepsilon=0} &= \langle (y' - y)^\wedge, \nabla_q f(t, \hat{y}) \rangle_{L^2} \\ &\stackrel{(4.7)}{=} (y' - y) \cdot (\nabla_q f(t, \hat{y}))^\vee. \end{aligned}$$

Hence, we can deduce the differentiability of g in y and

$$(4.12) \quad \nabla_y g(t, y) = (\nabla_q f(t, \hat{y}))^\vee.$$

Using the definition of t_* in (1.14), we can compute, for any $z, z' \in S_+^D$,

$$(4.13) \quad \left| (\nabla_q \psi(\hat{y} + t \nabla \xi(\hat{z})))^\vee - (\nabla_q \psi(\hat{y} + t \nabla \xi(\hat{z}'))^\vee \right| \leq t t_*^{-1} \|\hat{z} - \hat{z}'\|_{L^\infty} = t t_*^{-1} \|z - z'\|.$$

Due to $t < t_*$, by the Banach fixed point theorem, there is a unique $z \in S_+^D$ such that

$$(4.14) \quad z = (\nabla_q \psi(\hat{y} + t \nabla \xi(\hat{z})))^\vee.$$

Since $\hat{y} + t \nabla \xi(\hat{z})$ is a constant path, Lemma 4.7 implies that $\nabla_q \psi(\hat{y} + t \nabla \xi(\hat{z}))$ is also a constant path. Therefore, the above display yields

$$\hat{z} = \nabla_q \psi(\hat{y} + t \nabla \xi(\hat{z})).$$

On the other hand, (3.17) in Proposition 3.3 together with Lemma 3.4 implies that $\nabla_q f(t, \hat{y})$ is the unique solution in $\mathcal{Q}_2$ to $p = \nabla_q \psi(\hat{y} + t \nabla \xi(p))$. This, along with the above display, gives $\hat{z} = \nabla_q f(t, \hat{y})$. Hence, $\nabla_q f(t, \hat{y})$ is a constant path and thus $((\nabla_q f(t, \hat{y}))^\vee)^\wedge = \nabla_q f(t, \hat{y})$, which together with (4.12) gives the second relation in (4.10). Using these, we also have $\int \xi(\nabla_q f(t, \hat{y})) = \xi(\nabla_y g(t, y))$. This along with (1.34) that f satisfies the equation (3.1) implies that g satisfies the equation (4.9).

Lastly, we turn to (4.11) and the uniqueness statement above (4.11). Notice that $(y + t \nabla \xi(z))^\wedge = \hat{y} + t \nabla \xi(\hat{z})$. Due to (1.34), by the same argument for (4.12), we have $\nabla_y \phi(y + t \nabla \xi(z)) = (\nabla_q \psi(\hat{y} + t \nabla \xi(\hat{z})))^\vee$, which along with (4.14) gives $z = \nabla_y \phi(y + t \nabla \xi(z); x)$. Using (4.13), we can also see that z is the unique solution of this relation as claimed. Since we have shown $\hat{z} = \nabla_q f(t, \hat{y})$, by (3.18) (see $\mathcal{P}_{t,q}$ in (2.28)), we have $g(t, y) = f(t, \hat{y}) = \psi(\hat{y} + t \nabla \xi(\hat{z})) - t\theta(z)$, which along with (1.34) gives (4.11). $\square$

Proof of Theorem 1.9. The theorem follows from Proposition 4.8. $\square$

APPENDIX A. MULTI-SPECIES MODELS

We adapt the main results to multi-species spin glass models.

A.1. Setting. We fix a finite set $\mathcal{S}$ containing symbols for different species. For each $N \in \mathbb{N}$, let $(I_{N,s})_{s \in \mathcal{S}}$ be a partition of $\{1, \dots, N\}$. Each $I_{N,s}$ contains indices belonging to the s -species. For each $N \in \mathbb{N}$, we define

$$(A.1) \quad \lambda_{N,s} = |I_{N,s}|/N, \quad \forall s \in \mathcal{S}; \quad \lambda_N = (\lambda_{N,s})_{s \in \mathcal{S}}.$$

For each $N \in \mathbb{N} \cup \{\infty\}$, we consider the simplex

$$\blacktriangle_N = \left\{ (\lambda_s)_{s \in \mathcal{S}} \mid \lambda_s \in [0, 1] \cap \left(\frac{1}{N} \mathbb{Z} \right), \forall s \in \mathcal{S}; \sum_{s \in \mathcal{S}} \lambda_s = 1 \right\}.$$

Here $\frac{1}{N} \mathbb{Z} = \{k/N : k \in \mathbb{Z}\}$, with the understanding that $\mathbb{Z}/\infty = \mathbb{R}$ when $N = \infty$. In view of (A.1), we have $\lambda_N \in \blacktriangle_N$ for each $N \in \mathbb{N}$. Throughout this appendix, we assume that

$$(A.2) \quad \lim_{N \rightarrow \infty} \lambda_N = \lambda_\infty,$$

for some $\lambda_\infty \in \blacktriangle_N$.

For each $\mathbf{s} \in \mathcal{S}$, let $P_{\mathbf{s}}$ be a finite positive measure supported on $[-1, +1]$ and we assume $\text{supp } P_{\mathbf{s}} \not\supseteq \{0\}$ (related to the fact we require the support of P_1 to span $\mathbb{R}^D$). For every $N \in \mathbb{N}$, a spin configuration takes the form $\sigma = (\sigma_1, \dots, \sigma_N) \in [-1, +1]^N$, where each spin σ_n is independently drawn from $P_{\mathbf{s}}$ if $n \in I_{N,\mathbf{s}}$. In other words, denoting by P_N the distribution of σ , we have

$$dP_N(\sigma) = \otimes_{\mathbf{s} \in \mathcal{S}} \otimes_{i \in I_{N,\mathbf{s}}} dP_{\mathbf{s}}(\sigma_i).$$

For two spin configurations σ, σ' of size N and $\mathbf{s} \in \mathcal{S}$, we consider the overlap of the $\mathbf{s}$ -species:

$$R_{N,\mathbf{s}}(\sigma, \sigma') = \frac{1}{N} \sigma_{I_{N,\mathbf{s}}} \cdot \sigma'_{I_{N,\mathbf{s}}} \in [-1, +1],$$

where $\sigma_{I_{N,\mathbf{s}}} = (\sigma_i)_{i \in I_{N,\mathbf{s}}}$ is a vector in $\mathbb{R}^{I_{N,\mathbf{s}}}$ and similarly for $\sigma'_{I_{N,\mathbf{s}}}$. Hence, the spin overlap is $\mathbb{R}^{\mathcal{S}}$ -valued:

$$R_N(\sigma, \sigma') = (R_{N,\mathbf{s}}(\sigma, \sigma'))_{\mathbf{s} \in \mathcal{S}}.$$

Let $\xi : \mathbb{R}^{\mathcal{S}} \rightarrow \mathbb{R}$ be a smooth function and assume the existence of a centered Gaussian process $(H_N(\sigma))_{\sigma \in [-1, +1]^N}$ with covariance

$$\mathbb{E}[H_N(\sigma)H_N(\sigma')] = N\xi(R_N(\sigma, \sigma')).$$

Again, ξ is not assumed to be convex.

Fix any $\mathbf{s} \in \mathcal{S}$. Let $L_{\mathbf{s}} \in \mathbb{N}$ and let $(\chi_{\mathbf{s},i})_{i \in I_{N,\mathbf{s}}}$ be i.i.d. $\mathbb{R}^{L_{\mathbf{s}}}$ -valued random vectors, which are independent from other randomness. Let $d_{\mathbf{s}} \in \mathbb{N}$ and let $h_{\mathbf{s}} : \mathbb{R} \times \mathbb{R}^{L_{\mathbf{s}}} \rightarrow \mathbb{R}^{d_{\mathbf{s}}}$ be a bounded measurable function. For each $i \in I_{N,\mathbf{s}}$, we view $h_{\mathbf{s}}(\sigma_i, \chi_{\mathbf{s},i})$ as a generalized spin. We define the $\mathbf{s}$ -species mean magnetization and the total mean magnetization as, respectively,

$$m_{N,\mathbf{s}} = \frac{1}{N} \sum_{i \in I_{N,\mathbf{s}}} h_{\mathbf{s}}(\sigma_i, \chi_i) \quad \text{and} \quad m_N = (m_{N,\mathbf{s}})_{\mathbf{s} \in \mathcal{S}},$$

which are $\mathbb{R}^{d_{\mathbf{s}}}$ -valued and $\prod_{\mathbf{s} \in \mathcal{S}} \mathbb{R}^{d_{\mathbf{s}}}$ -valued. Let $G : \prod_{\mathbf{s} \in \mathcal{S}} \mathbb{R}^{d_{\mathbf{s}}} \rightarrow \mathbb{R}$ be a continuous function and we add to the system the Curie–Weiss-type interaction $NG(m_N)$.

Recall that in the setting of the vector spin glass, $\mathcal{Q}$ is the collection of increasing càdlàg paths $q : [0, 1) \rightarrow S_+^D$. Here, we take $\mathcal{Q}(1)$ to be $\mathcal{Q}$ with $D = 1$. In other words, $\mathcal{Q}(1)$ consists of all increasing càdlàg paths from $[0, 1)$ to $\mathbb{R}_+$. Here, 1 indicates that the paths are one-dimensional. Then, for $p \in [0, \infty]$, we write $\mathcal{Q}_p(1) = \mathcal{Q}(1) \cap L^p([0, 1); \mathbb{R})$. For each $q \in \mathcal{Q}_\infty(1)$, we set $q(1) = \lim_{s \uparrow 1} q(s)$. The collection of paths relevant in the multi-species setting is

$$\mathcal{Q}_p^{\mathcal{S}} = \prod_{\mathbf{s} \in \mathcal{S}} \mathcal{Q}_p(1).$$

For each $q \in \mathcal{Q}_p^{\mathcal{S}}$, we write $q = (q_{\mathbf{s}})_{\mathbf{s} \in \mathcal{S}}$ with each $q_{\mathbf{s}} \in \mathcal{Q}_p(1)$.

For $\mathbf{s} \in \mathcal{S}$ and $q_{\mathbf{s}} \in \mathcal{Q}_{\infty}(1)$, let $(w_i^{q_{\mathbf{s}}})$ be i.i.d. centered Gaussian fields where the covariance of each $(w_i^{q_{\mathbf{s}}}(\alpha))_{\alpha \in \text{supp } \mathfrak{R}}$ is given similarly as in (1.9). For $N \in \mathbb{N}$, $t \in \mathbb{R}_+$, $q \in \mathcal{Q}_{\infty}^{\mathcal{S}}$, we consider the Hamiltonian

$$\begin{aligned} H_{N;t,q}^G(\sigma, \alpha) &= \sqrt{2t}H_N(\sigma) - tN\xi(R_N(\sigma, \sigma')) \\ &\quad + \sqrt{2} \sum_{\mathbf{s} \in \mathcal{S}} \sum_{i \in I_{N,\mathbf{s}}} w_i^{q_{\mathbf{s}}}(\alpha) \sigma_i - q(1) \cdot R_N(\sigma, \sigma') \\ &\quad + NG(m_N). \end{aligned}$$

Then, we consider the enriched free energy defined by

$$(A.3) \quad \overline{F}_N^G(t, q) = -\frac{1}{N} \mathbb{E} \log \iint \exp(H_{N;t,q}^G(\sigma, \alpha)) dP_N(\sigma) d\mathfrak{R}(\alpha)$$

where $\mathbb{E}$ averages all the randomness.

For $q = (q_{\mathbf{s}})_{\mathbf{s} \in \mathcal{S}} \in \mathcal{Q}_{\infty}^{\mathcal{S}}$, $x = (x_{\mathbf{s}})_{\mathbf{s} \in \mathcal{S}} \in \prod_{\mathbf{s} \in \mathcal{S}} \mathbb{R}^{d_{\mathbf{s}}}$, and $\lambda \in \blacktriangle_{\infty}$, we set

$$\begin{aligned} \psi_{\mathbf{s}}(q_{\mathbf{s}}; x_{\mathbf{s}}) &= -\mathbb{E} \log \iint \exp(\sqrt{2}w^{q_{\mathbf{s}}}(\alpha)\tau - q_{\mathbf{s}}(1)\tau^2 + x_{\mathbf{s}} \cdot h_{\mathbf{s}}(\tau, \chi_{\mathbf{s},1})) dP_{\mathbf{s}}(\tau) d\mathfrak{R}(\alpha) \end{aligned}$$

and

$$\psi_{\lambda}(q; x) = \sum_{\mathbf{s} \in \mathcal{S}} \lambda_{\mathbf{s}} \psi_{\mathbf{s}}(q_{\mathbf{s}}; x_{\mathbf{s}}).$$

Given ξ , we define

$$\theta(a) = a \cdot \nabla \xi(a) - \xi(a), \quad \forall a \in \mathbb{R}^{\mathcal{S}}.$$

A.2. Adapted main results. Theorems 1.4, 1.8, and 1.9 can be adapted as follows in this setting. We define $t_{\star}$ by

$$(A.4) \quad t_{\star}^{-1} = \sup_x \|\nabla_q \psi_{\lambda_{\infty}}(\cdot; x)\|_{\text{Lip}(\mathcal{Q}_{\infty}^{\mathcal{S}}, \mathcal{Q}_{\infty}^{\mathcal{S}})} \|\nabla \xi|_{[-1,1]^{\mathcal{S}}}\|_{\text{Lip}([-1,1]^{\mathcal{S}}; \mathbb{R}^{\mathcal{S}})}.$$

Theorem A.1 (Limit free energy). *Assume (A.2). For $x \in \prod_{\mathbf{s} \in \mathcal{S}} \mathbb{R}^{d_{\mathbf{s}}}$, there is a twice Fréchet differentiable function $f(\cdot, \cdot; x) : [0, t_{\star}) \times \mathcal{Q}_{\infty}^{\mathcal{S}} \rightarrow \mathbb{R}$ that satisfies*

$$\begin{cases} \partial_t f(t, q; x) - \int \xi(\nabla_q f(t, q; x)) = 0 & \text{for } (t, q) \in (0, t_{\star}) \times \mathcal{Q}_{\infty}^{\mathcal{S}}, \\ f(0, q; x) = \psi_{\lambda_{\infty}}(q; x) & \text{for } q \in \mathcal{Q}_{\infty}^{\mathcal{S}}. \end{cases}$$

For every $(t, q) \in [0, t_{\star}) \times \mathcal{Q}_{\infty}^{\mathcal{S}}$ and every continuous function $G : \prod_{\mathbf{s} \in \mathcal{S}} \mathbb{R}^{d_{\mathbf{s}}} \rightarrow \mathbb{R}$ we have

$$\lim_{N \rightarrow +\infty} \overline{F}_N^G(t, q) = \inf_m \sup_x \{f(t, q; x) + m \cdot x - G(m)\}.$$

Moreover, for every $x \in \prod_{\mathbf{s} \in \mathcal{S}} \mathbb{R}^{d_{\mathbf{s}}}$, there is a unique solution $p^x \in \mathcal{Q}_{\infty}^{\mathcal{S}}$ of the fixed point equation

$$p = \nabla_q \psi_{\lambda_{\infty}}(q + t \nabla \xi(p); x) \quad \text{for } p \in \mathcal{Q}_{\infty}^{\mathcal{S}}.$$

We have,

$$f(t, q; x) = \psi_{\lambda_\infty}(q + t\nabla\xi(p^x); x) - t \int_0^1 \theta(p^x(u)) du \quad \text{and} \quad \nabla_q f(t, q; x) = p^x.$$

The Gibbs measure associated to $H_{N;t,q}^G$ is the (random) probability measure defined on $R^N \times \text{supp } \mathfrak{R}$ by

$$\langle \cdot \rangle_{N;t,q}^G \propto \exp(H_{N;t,q}^G(\sigma, \alpha)) dP_N(\sigma) \mathfrak{R}(\alpha).$$

We denote by Ω the underlying probability space for $H_{N;t,q}^G$ and $\mathfrak{R}$, and write $\langle \cdot \rangle_{N;t,q}^{G,\omega}$ as a realization of $\langle \cdot \rangle_{N;t,q}^G$ for any $\omega \in \Omega$.

Theorem A.2 (Large deviations of m_N). *Assume (A.2). Let $t < t_*$ and $q \in \mathcal{Q}_\infty^\mathcal{J}$. There is a full measure subset $\Omega' \subseteq \Omega$ such that the following holds. For every $\omega \in \Omega'$ and every continuous function $G : \prod_{s \in \mathcal{S}} \mathbb{R}^{d_s} \rightarrow \mathbb{R}$, the random variable m_N satisfies a large deviation principle under $\langle \cdot \rangle_{N;t,q}^{G,\omega}$ with rate function $I_{t,q}^G$ defined by*

$$I_{t,q}^G(m) = -G(m) + f^*(t, q; m) + \sup_{m'} \{G(m') - f(t, q; m')\}, \quad \forall m \in \prod_{s \in \mathcal{S}} \mathbb{R}^{d_s},$$

where $f^*(t, q; m) = \sup_x \{x \cdot m + f(t, q; x)\}$ with f given in Theorem A.1.

Given $y \in \mathbb{R}_+^\mathcal{J}$, we define $\hat{y}$ similar as in (1.25) to be a constant path. For $y = (y_s)_{s \in \mathcal{S}} \in \mathbb{R}_+^\mathcal{J}$, $x = (x_s)_{s \in \mathcal{S}} \in \prod_{s \in \mathcal{S}} \mathbb{R}^{d_s}$, and $\lambda \in \blacktriangle_\infty$, we set

$$\phi_s(y_s; x_s) = -\mathbb{E} \log \int \exp(\sqrt{2}y_s \eta_s \tau - y_s \tau^2 + x_s \cdot h_s(\tau, \chi_{s,1})) dP_s(\tau),$$

where η_s is an independent standard Gaussian random variable, and we also set

$$\phi_\lambda(y; x) = \sum_{s \in \mathcal{S}} \lambda_s \phi_s(y_s; x_s).$$

By a similar argument as in Remark 1.10, we have $\phi_\lambda(y; x) = \psi_\lambda(\hat{y}; x)$.

Theorem A.3 (Replica symmetric form). *Assume (A.2). Let $x \in \prod_{s \in \mathcal{S}} \mathbb{R}^{d_s}$. Define $g(\cdot, \cdot; x) : [0, t_*) \times \mathbb{R}_+^\mathcal{J} \rightarrow \mathbb{R}$ by*

$$g(t, y; x) = f(t, \hat{y}; x),$$

with f given in Theorem A.1. The function $g(\cdot, \cdot; x)$ is differentiable on $[0, t_*) \times \mathbb{R}_+^\mathcal{J}$ and satisfies

$$\begin{cases} \partial_t g(t, y; x) - \xi(\nabla_y g(t, y; x)) = 0 & \text{for } (t, y) \in (0, t_*) \times \mathbb{R}_+^\mathcal{J}, \\ g(0, y; x) = \phi_{\lambda_\infty}(y; x) & \text{for } y \in \mathbb{R}_+^\mathcal{J}. \end{cases}$$

Moreover, at every $(t, y) \in (0, t_*) \times \mathbb{R}_+^\mathcal{J}$, there is a unique solution $z \in \mathbb{R}_+^\mathcal{J}$ to the fixed point equation

$$z' = \nabla_y \phi_{\lambda_\infty}(t\nabla\xi(z'); x) \quad \text{for } z' \in \mathbb{R}_+^\mathcal{J},$$

and we have

$$g(t, y; x) = \phi_{\lambda_\infty}(y + t\nabla\xi(z); x) - t\theta(z) \quad \text{and} \quad \nabla_y g(t, y; x) = z.$$

A.3. Recipe for adaptation. Instead of rewriting everything in detail, we want to avoid repetition and give a recipe for adapting existing results to the multi-species setting.

We mention the difference in initial conditions. In the vector spin glass setting, we simply have $\overline{F}_N^{x \mapsto x \cdot m}(0, q) = \psi(q; x)$ (see the line above (1.12)). Here, we have (see [19, Lemma 4.11]) $\overline{F}_N^{x \mapsto x \cdot m}(0, q) = \psi_{\lambda_N}(q; x)$ and thus, under the assumption that λ_N converges to λ_∞ ,

$$\lim_{N \rightarrow \infty} \overline{F}_N^{x \mapsto x \cdot m}(0, q) = \psi_{\lambda_\infty}(q; x).$$

In the current setting, ψ_s has the same form as ψ in (1.12) in the vector spin glass. So, each ψ_s satisfies the properties in Proposition 2.2 with $\mathcal{Q}_p$ therein replaced by $\mathcal{Q}_p(1)$.

To transfer these properties to ψ_{λ_∞} , we first recall the norms on $\mathcal{Q}_p^\mathcal{S}$. We view $\mathcal{Q}_p^\mathcal{S}$ as embedded into $L^p([0, 1]; \mathbb{R}^\mathcal{S})$, equipped with the norm

$$|\kappa|_{L^p} = \left(\int_0^1 \left(\sum_{s \in \mathcal{S}} \kappa_s(r)^2 \right)^{\frac{p}{2}} dr \right)^{\frac{1}{p}}.$$

Observe that at $p = 2$, we have $|\kappa|_{L^2}^2 = \sum_{s \in \mathcal{S}} |\kappa_s|_{L^2}^2$.

One can verify that Proposition 3.3, Corollary 4.3, and Lemma 4.4 hold with $t_\star$ defined in (A.4). Indeed, checking the proofs, one can see that the definition of $t_\star$ (previously in (1.14)) are used in (3.8), the line below (3.14), (3.24), and (4.3). Then, one can verify that the definition in (A.4) still works in all these cases. To restate and reprove these four propositions, one needs to replace $\mathcal{Q}_p$, ψ (see (1.12)), S_+^D , and $\mathbb{R}^d$ therein by $\mathcal{Q}_p^\mathcal{S}$, ψ_{λ_∞} , $\mathbb{R}_+^\mathcal{S}$, and $\prod_{s \in \mathcal{S}} \mathbb{R}^{d_s}$, respectively. Cited results from [22] for vector spin glasses can be replaced by counterparts in [19] for multi-species spin glasses.

Once those three results are proven in the setting of multi-species models, one can straightforwardly use the same arguments as in the rest of Section 4 to finish the proofs of Theorems A.1, A.2, and A.3.

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