TD2: Poisson Processes

Exercice 1 — *Jump times of a Poisson process.*
Let $\lambda > 0$ consider a Poisson process X with intensity λ, let $(J_n)_n$ be the jump times of X. Let $n \geq 1$ and $t > 0$, let $U_1, \dots, U_n$ be independent uniform random variables in $[0, t]$. Let σ be the (random) permutation of $\{1, \dots, n\}$ such that,

$$U_{\sigma(1)} \leq U_{\sigma(2)} \leq \cdots \leq U_{\sigma(n)}.$$

(1) Prove that σ is well-defined on a set of measure 1.
If the $(U_i)_i$ are pairwise distinct, then σ is well-defined. We have,

$$\mathbb{P}\left(\exists i \neq j, U_i = U_j\right) \leq \sum_{i \neq j} \mathbb{P}(U_i = U_j) = 0.$$

So, almost surely the $(U_i)_i$ are pairwise disjoint and σ is well-defined on a set of measure 1.

(2) Show that the random variable $(U_{\sigma(i)})_{1 \leq i \leq n}$ has density

$$d_n(u_1, \dots, u_n) = n!/t^n \mathbf{1}\{u_1 < \cdots < u_n\}.$$

We have for every bounded measurable $f : [0, t]^n \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}[f(U_{\sigma(1)}, \dots, U_{\sigma(n)})] &= \sum_{\sigma_0 \in S_n} \mathbb{E}[\mathbf{1}_{\sigma = \sigma_0} f(U_{\sigma_0(1)}, \dots, U_{\sigma_0(n)})] \\
&= \sum_{\sigma_0 \in S_n} \mathbb{E}[\mathbf{1}_{U_{\sigma_0(1)} < \cdots < U_{\sigma_0(n)}} f(U_{\sigma_0(1)}, \dots, U_{\sigma_0(n)})] \\
&= \sum_{\sigma_0 \in S_n} \mathbb{E}[\mathbf{1}_{U_1 < \cdots < U_n} f(U_1, \dots, U_n)] \\
&= n!\, \mathbb{E}[\mathbf{1}_{U_1 < \cdots < U_n} f(U_1, \dots, U_n)] \\
&= \int_{[0,t]^n} n! \mathbf{1}_{u_1 < \cdots < u_n} f(u_1, \dots, u_n) \frac{du_1}{t} \dots \frac{du_n}{t} \\
&= \int_{[0,t]^n} f(u_1, \dots, u_n) d_n(u) du_1 \dots du_n.
\end{aligned}$$

Where the manipulation in the previous display comes from the fact for every fixed $\sigma_0 \in S_n$, $(U_{\sigma_0(i)})_i$ and $(U_i)_i$ have the same law.

(3) Show that the density of $(J_1, \dots, J_n)$ conditionally on $\{X_t = n\}$ is d_n, that is for any non-negative measurable function $f : \mathbb{R}^n \to \mathbb{R}_+$, we have

$$\mathbb{E}\left[f(J_1, \dots, J_n) | X_t = n\right] = \int_{[0,t]^n} f(s_1, \dots, s_n) d_n(s_1, \dots, s_n) ds_1 \dots ds_n.$$

A Poisson process almost surely doesn't blow up, in particular almost surely for every $n \geq 1$, $S_n = J_n - J_{n-1}$ is well-defined and finite. Furthermore, the S_n's are iid and follow an exponential law of parameter λ. For every $n \geq 1$ the density of $(S_1, \dots, S_{n+1})$ is $\lambda^{n+1} \exp\left(-\sum_{i=1}^{n+1} \lambda s_i\right) \mathbf{1}_{\forall i, s_i \geq 0}$. Therefore, $(J_1, \dots, J_{n+1})$ has density $\lambda^{n+1} \exp\left(-\lambda j_{n+1}\right) \mathbf{1}_{j_1 \leq \dots \leq j_{n+1}}$. Thus,

$$\begin{aligned}
\mathbb{E}[f(J_1, \dots, J_n)\mathbf{1}_{X_t = n}] &= \mathbb{E}[f(J_1, \dots, J_n)\mathbf{1}_{J_n \leq t \leq J_{n+1}}] \\
&= \int_{\mathbb{R}_+^{n+1}} f(j_1, \dots, j_n)\mathbf{1}_{j_1 \leq \dots \leq j_n \leq t \leq j_{n+1}} \lambda^{n+1} \exp\left(-\lambda j_{n+1}\right) dj_1 \dots dj_{n+1} \\
&= \int_{[0,t]^n} f(j_1, \dots, j_n)\mathbf{1}_{j_1 \leq \dots \leq j_n} dj_1 \dots dj_n \times \int_t^{+\infty} \lambda^{n+1} \exp\left(-\lambda j_{n+1}\right) dj_{n+1} \\
&= \int_{[0,t]^n} f(j_1, \dots, j_n)\mathbf{1}_{j_1 \leq \dots \leq j_n} dj_1 \dots dj_n \lambda^n e^{-\lambda t}.
\end{aligned}$$

Furthermore, X_t follows a Poisson law of parameter λt so $\mathbb{P}(X_t = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$. So,

$$\mathbb{E}[f(J_1, \dots, J_n) | X_t = n] = \int_{[0,t]^n} f(j_1, \dots, j_n)\mathbf{1}_{j_1 \leq \dots \leq j_n} \frac{n!}{t^n} dj_1 \dots dj_n.$$

(4) Deduce from the previous questions a way to sample the trajectories on $[0, t]$ of a Poisson process using only uniform random variables and Poisson random variables.
Sample N a random Poisson random variable of parameter λt, and $(U_i)_{i \geq 1}$ independent uniform random variables. Take the N first uniform random variables $U_1, \dots, U_N$ and order them as $U_{\sigma(1)} < \dots < U_{\sigma(N)}$. One can then define for every $i \in \{1, \dots, N-1\}$ and $s \in [U_{\sigma(i)}, U_{\sigma(i+1)})$, $X_s = i$ and for $s \in [U_{\sigma(N)}, t]$ $X_s = N$. The process $(X_s)_{s \in [0,t]}$ has the law of Poisson process of intensity $\lambda > 0$ on $[0, t]$.

Exercice 2 — *$M/GI/\infty$ queue.*

Let $X = (X_t)_{t \geq 0}$ be a Poisson process of intensity $\lambda > 0$, we denote $(J_n)_n$ the jump times of X. Let $(Z_n)_n$ be iid random variables, we denote G the cdf of Z_1 and $1/\mu$ the mean of Z_1. Consider the following model, you operate a restaurant in which the n^{th} customer arrives at time J_n and leaves at time $J_n + Z_n$. You want to estimate the number N_t of customers in the shop at time t. Note that for every $t \geq 0$, we have

$$N_t = \sum_n \mathbf{1}\{J_n \leq t \leq J_n + Z_n\}.$$

(1) Let X a Poisson random variable of parameter $\alpha > 0$ and $(B_n)_n$ be iid Bernoulli random variables of parameter p independent from X, show that $Y = \sum_{n=1}^{X} B_n$ is a Poisson random variable of parameter αp.

Let $N \geq 0$, conditionally on $X = N$, Y is binomial of parameter (N, p). So for every $n \in \{0, \ldots, N\}$,

$$\mathbb{P}(Y = n, X = N) = e^{-\alpha}\frac{\alpha^N}{N!}\binom{N}{n}p^n(1-p)^{N-n}.$$

Now fix $n \geq 0$, summing the above display over $N \geq 0$, obtain

$$\mathbb{P}(Y = n) = e^{-\alpha p}\frac{(\alpha p)^n}{n!}.$$

(2) Let $t \geq 0$, $n \geq 0$ and let U denote a uniform random variable in $[0, t]$, define $p = \mathbb{P}(Z_1 > U)$. Show that conditionally on $X_t = n$, the random variable N_t is Binomial random variable of parameter (n, p).
Let $U_1, \ldots, U_n$ be independent uniform random variables in $[0, t]$ and let σ be the permutation of $\{1, \ldots, n\}$ defined almost surely by $U_{\sigma(1)} < \cdots < U_{\sigma(n)}$. Conditionally on $X_t = n$, $(J_1, \ldots, J_n)$ has the law of $(U_{\sigma(1)}, \ldots, U_{\sigma(n)})$. Thus, still conditionally on $X_t = n$, we have

$$\begin{aligned} N_t &= \sum_{k=1}^{n} \mathbf{1}\{t \leq J_k + Z_k\} \\ &\overset{(d)}{=} \sum_{k=1}^{n} \mathbf{1}\{t \leq U_{\sigma(k)} + Z_k\} \\ &\overset{(d)}{=} \sum_{k=1}^{n} \mathbf{1}\{t \leq U_k + Z_k\}. \end{aligned}$$

So conditionally on $X_t = n$, N_t is a binomial random variable of parameter (n, p) where $p = \mathbb{P}(t \leq Z_1 + U) = \mathbb{P}(t - U \leq Z_1) = \mathbb{P}(U \leq Z_1)$. The last inequality following from the fact that $t - U \overset{(d)}{=} U$.

(3) Let $t > 0$ and $\alpha(t) = \lambda \int_0^t \mathbb{P}(Z_1 > x)dx$, show that $N(t)$ is a Poisson random variable with parameter $\alpha(t)$.
According to the previous given $k \geq 0$ and $n \geq k$, we have

$$\mathbb{P}(N_t = k, X_t = n) = e^{-\lambda t}(\lambda t)^n/n!\binom{n}{k}p^k(1-p)^{n-k}.$$

Summing over $n \geq k$, we obtain

$$\mathbb{P}(N_t = k) = e^{-\lambda t p}(\lambda t p)^k/k!.$$

This means that N_t is a Poisson random variable of parameter $\lambda t p$, finally $p = \mathbb{P}(Z_1 > U) = \frac{1}{t}\int_0^t \mathbb{P}(Z_1 > s)ds$, thus $\lambda t p = \alpha(t)$.

(4) Show that as $t \to \infty$, N_t converges in law toward a Poisson law of parameter $\rho = \lambda/\mu$.

As $t \to \infty$, we have $\alpha(t) \to \lambda \int_0^\infty \mathbb{P}(Z_1 > s)ds = \lambda \mathbb{E} S = \lambda/\mu$. It follows that,

$$\lim_{t\to\infty} \mathbb{P}(N_t = k) = e^{-\rho}\frac{\rho^k}{k!}.$$

In France approximately, 1903896 new cars have been bought each year between 1967 and 2023 (source : CCFA, Comité des Constructeurs Français d'Automobiles). Assume that the French people buy cars according to a Poisson Process of parameter $\lambda = 1903896$ per year and that there was no car bought before 1967.

(5) Assume that each car owner keeps its car for a duration uniform between 0 and 20 years. What is the expected number of cars in the French fleet in the year 1977 ? what about in the year 1987 ? and Afterward ?

If we keep the notations of the previous section, the expected number of cars in the fleet in the year $1967+t$ is $\alpha(t) = \lambda \int_0^t \mathbb{P}(Z_1 > s)ds$. If Z_1 follows a uniform random variable in $[0, b]$ with $b = 20$, we have for every $s < b$, $\mathbb{P}(Z_1 > s)ds = 1 - s/b$ and $= 0$ otherwise. It follows that α is constant after $t = b$ and for $t \leq b$, we have,

$$\alpha(t) = \lambda \int_0^t 1 - \frac{s}{b} ds = \lambda(t - \frac{t^2}{2b}).$$

We obtain,

$$\alpha(10) = \frac{3}{8}\lambda b \simeq 15M$$
$$\alpha(20) = \lambda b/2 \simeq 20M$$

(6) Answer the previous question now assuming that each owner keeps its car for an exponential duration of parameter 1/10.

Similar computations yield,

$$\alpha(t) = \lambda \int_0^t \exp(-2s/b)ds = \frac{\lambda b}{2}\left(1 - e^{-2t/b}\right).$$