TD8 : Construction of the Brownian Motion

Exercise 1 — *Transformations.*
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(1) Show that for any $\lambda \in \mathbb{R}_+^\star$, the process $(\lambda^{-1/2}B_{\lambda t})_{t\geq 0}$ is a Brownian motion.
(2) Show that $B_1 - B_{1-t}$ is a Brownian motion on $[0,1]$.

We first consider the finite-dimensional marginals of the new process $(X_t)_t$ in these two cases. Remark at first that they still form centered Gaussian vectors, since they are each obtained by a very simple linear transform of some f.d.m. of B. Now we only need to compute covariances.

(1) $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(\lambda^{-1/2}B_{\lambda s}, \lambda^{-1/2}B_{\lambda t}) = \lambda^{-1}\mathrm{Cov}(B_{\lambda s}, B_{\lambda t}) = \lambda^{-1}(\lambda s \wedge \lambda t) = s \wedge t$.
(2) For $0 \leq s,t \leq 1$, $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(B_1 - B_{1-s}, B_1 - B_{1-t}) = \mathrm{Cov}(B_1, B_1) - \mathrm{Cov}(B_1, B_{1-t}) - \mathrm{Cov}(B_{1-s}, B_1) + \mathrm{Cov}(B_{1-s}, B_{1-t}) = 1 - (1-t) - (1-s) + (1-s) \wedge (1-t) = 1 + (t-1) \wedge (s-1) = t \wedge s$.

Now since those processes are continuous on their domain of definition, they are Brownian motions.

Exercise 2 — *Constructing a Brownian motion indexed by* $\mathbb{R}_+$.
Let $(B^{(n)})_n$ be a sequence of independent Brownian motions defined on $[0,1]$. For every $t \geq 0$, define

$$B_t = B^{(\lfloor t \rfloor)}_{t - \lfloor t \rfloor} + \sum_{i=0}^{\lfloor t \rfloor - 1} B^{(i)}_1.$$

Show that $(B_t)_{t\geq 0}$ is a Brownian motion.
We can check continuity for all ω manually. Now a f.d.m. $B_{t_1}, \dots, B_{t_k}$ is a very simple linear transform of (some f.d.m. of $B^{(1)}$, some f.d.m. of $B^{(2)}$,..., some f.d.m. of $B^{(\lfloor t_k \rfloor)}$). Because of the independence assumption, this is a big Gaussian vector. Now we compute covariances. Let $s \leq t$.

$$\begin{aligned}\mathrm{Cov}(B_s, B_t) &= \mathrm{Cov}\left(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor} + \sum_{i=0}^{\lfloor s \rfloor - 1} B^{(i)}_1, B^{(\lfloor \rfloor)}_{t-\lfloor t \rfloor} + \sum_{i=0}^{\lfloor t \rfloor - 1} B^{(i)}_1\right) \\ &= \begin{cases} \sum_{i=0}^{\lfloor s \rfloor - 1} \mathrm{Var}(B^{(i)}_1) + \mathrm{Cov}(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor}, B^{(\lfloor t \rfloor)}_{t-\lfloor t \rfloor}) \text{ if } \lfloor t \rfloor = \lfloor s \rfloor \\ \sum_{i=0}^{\lfloor s \rfloor - 1} \mathrm{Var}(B^{(i)}_1) + \mathrm{Cov}(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor}, B^{(\lfloor s \rfloor)}_1) \text{ if } \lfloor t \rfloor > \lfloor s \rfloor \end{cases} \\ &= s.\end{aligned}$$

This completes the proof.

Exercise 3 — *Lévy's construction of the Brownian motion.*
Let $H = L^2([0,1])$ with the usual inner product. For $t \geq 0$ let $I_t = \mathbb{1}_{[0,t]} \in H$. We also set $(e_i)_{i\in\mathbb{N}}$ to be an orthonormal basis of H.

(1) Check that $\langle I_s, I_t\rangle = s \wedge t$.

Immediate.

(2) Assume that there exists a H-valued standard Gaussian random variable. That is, a random variable $\xi \in H$, such that for every $x \in H$, $\langle x, \xi\rangle \sim \mathcal{N}(0, |x|^2)$.

(a) Using the random variable ξ and the functions $(I_t)_{t\geq 0}$, build a Gaussian process $(B_t)_{t\in[0,1]}$ such that $\mathrm{Cov}(B_s, B_t) = s \wedge t$.
Setting $B_t = \langle \xi, I_t\rangle$ yields a Gaussian process with the right covariance kernel. It can be checked by defintion of ξ that every linear combination of coordinates of $(B_{t_1}, \dots, B_{t_k})$ is Gaussian and that $\mathbb{E}[(B_t - B_s)^2] = |t - s|$.

(b) Let $Z_i = \langle \xi, e_i\rangle$, so that $\xi = \sum_{i\in\mathbb{N}} Z_i e_i$. Show that the (Z_i) are independent standard Gaussians (*Hint:* Compute the characteristic function of finite subvectors.). Deduce that the process of the previous question would satisfy,

$$B_t = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s)ds. \tag{†}$$

We have,

$$\mathbb{E}[\exp(it_1 Z_{i_1} + \cdots + it_p Z_{i_p})] = \mathbb{E}[\exp(i\langle t_1 e_1 + \cdots + t_p e_p, \xi\rangle] = \prod_{i=1}^{p} e^{-it_p^2/2}.$$

Hence the distribution is that of i.i.d. standard Gaussians.

$$B_t = \langle \xi, I_t\rangle = \sum_{i=0}^{\infty} \langle \xi, e_i\rangle\langle I_t, e_i\rangle = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s)ds. \tag{†}$$

(c) By computing $|\xi|^2$, show that ξ cannot exist.
$\|\xi\|^2 = \sum_{i=0}^{\infty} Z_i^2$ which is a.s. not convergent. Indedd, $p = \mathbb{P}(Z_n \geq 1)$ is independent of n and > 0 so $\sum_n \mathbb{P}(Z_n \geq 1)$ diverges and by Borel-Cantelli

$$\mathbb{P}(\forall N \exists n \geq N, Z_n \geq 1) = 1.$$

This last event is contained in the event $\{\sum_{i=0}^{\infty} Z_i^2 \text{ diverges}\}$.

(3) Define $h_0 = 0$ and for $n \geq 0$ and $0 \leq k < 2^n$,

$$h_{k,n} := 2^{n/2}\left(\mathbb{1}_{\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right]} - \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]}\right),$$

We admit (or recall) that $(h_{k,n})_{k,n}$ is an orthonormal basis of H called the Haar wavelet basis. Let $(Z_{n,k})_{n,k}$ be a family of independent standard Gaussian random variables. For every $t \geq 0$ set

$$B_t = tZ + \sum_{n=0}^{\infty} F_n(t), \tag{††}$$

where $F_n(t) = \sum_{k=0}^{2^n-1} Z_{n,k} f_{n,k}(t)$ and $f_{n,k}(t) = \int_0^t h_{n,k}(s)ds$.

(a) Using the inequality $\mathbb{P}(|Y| \geq \lambda) \leq \frac{\sqrt{2/\pi}}{\lambda} e^{-\lambda^2}$ for $\lambda > 0$ and $Y \sim \mathcal{N}(0,1)$, show that

$$\mathbb{P}\left(2^{-\frac{n+2}{2}} \max_{0\leq k<2^n} |Z_{n,k}| > \frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right).$$

$$\mathbb{P}\left(2^{-\frac{n+2}{2}} \max_{0\leq k<2^n} |Z_{n,k}| > \frac{1}{n^2}\right) \leq 2^n \mathbb{P}\left(2^{-\frac{n+2}{2}} |Z_{n,1}| > \frac{1}{n^2}\right) \leq \frac{n^2}{\sqrt{\pi}} 2^{\frac{n-3}{2}} \exp\left(-\frac{2^{n+1}}{n^4}\right) = o\left(\frac{1}{n^2}\right).$$

(b) Show that $\mathbb{P}\left(\|F_n\|_\infty \leq \frac{1}{n^2} \text{ for } n \text{ large enough }\right) = 1$.
We have $\|F_n\|_\infty \leq 2^{-\frac{n+2}{2}} \max_{0\leq k<2^n} |Z_{n,k}|$, by the previous question $\sum_n \mathbb{P}\left(\|F_n\|_\infty > \frac{1}{n^2}\right) \leq +\infty$ so by Borel-Cantelli, almost surely for n large enough $\|F_n\|_\infty \leq 1/n^2$.

(c) Show that almost surely, the sum of functions in (††) converges uniformly on $[0,1]$ to a (random) continuous function.
The result follows directly from the previous question by looking at the difference between partial sums nad using Cauhchy's criterion.

(4) ($\star$) Prove the same result than in the previous question when we use the Fourier basis $e_0 = 1$, and $e_m(t) = \sqrt{2}\cos(\pi m t)$ in (†) rather than the Haar wavelet basis.

Exercise 4 — *Time inversion.*
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Set $X_t = tB_{1/t}$ for $t > 0$ and $X_0 = 0$.

(1) Show that X has the finite-dimensional marginals of a Brownian motion.
We first consider the finite-dimensional marginals of the new process $(X_t)_t$ in this case. Remark at first that they still form centered Gaussian vectors, since they are each obtained by a very simple linear transform of some f.d.m. of B. Now we only need to compute covariances: If $0 < s, t$, $\text{Cov}(X_s, X_t) = \text{Cov}(sB_{1/s}, tB_{1/t}) = st(s^{-1} \wedge t^{-1}) = t \wedge s$. If either $t = 0$ or $s = 0$, then we get $0 = s \wedge t$ for the covariance too.

(2) Show that the set $U = \{f \in \mathbb{R}^{\mathbb{Q}_+}, \lim_{t\to 0, t\in\mathbb{Q}} f_t = 0\} \subset \mathbb{R}^{\mathbb{Q}_+}$ is measurable.
Observe that,

$$U = \bigcap_{n\geq 1} \bigcup_{m\in\mathbb{N}} \bigcap_{\substack{q\in\mathbb{Q}_+ \\ q\leq 1/m}} \{A : |f_q| < 1/n\},$$

hence U belongs to the σ-algebra generated by finite-dimensional sets.

(3) Deduce that $(X_t)_t$ is continuous almost surely, hence may be modified on a negligible event to form a Brownian motion.
By the $\pi-\lambda$ (monotone class) theorem, two measures that coincide on a π-system Π (a family of sets stable by finite intersection), coincide on the generated σ-algebra $\sigma(\Pi)$. As a result, since B and X have the same finite-dimensional marginals, then

$$\mathbb{P}(X_{|\mathbb{Q}_+} \in U) = \mathbb{P}(B_{|\mathbb{Q}_+} \in U) = 1.$$

Hence we have with probability one that:
(a) $t \mapsto X_t$ is continuous on $(0, \infty)$,
(b) $X_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}]{} X_0$

wich together implies continuity on the whole of $[0, \infty)$. Now if we change the X to the constant zero function whenever X is not continuous, this makes X continuous for all ω without changing the f.d.m's. So X is a Brownian motion.